Private independence testing across two parties

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Abstract

We introduce $\pi$-test, a privacy-preserving algorithm for testing statistical independence between data distributed across multiple parties. Our algorithm relies on privately estimating the distance correlation between datasets, a quantitative measure of independence introduced in Székely et al. [2007]. We establish both additive and multiplicative error bounds on the utility of our differentially private test, which we believe will find applications in a variety of distributed hypothesis testing settings involving sensitive data.

1 Introduction

Hypothesis testing is a staple of theoretical and applied statistics, whose impact in decision making has gone beyond that of statistics into a wide-range of adjacent and seemingly far-flung fields of science and engineering. At a high level, hypothesis testing formalizes the choice to reject (or not reject) a null hypothesis $H_0$ in comparison to an alternate hypothesis $H_1$, where both hypotheses are defined as disjoint sets of probability distributions. The decision is to be made with a high probability upon looking at samples from some unknown distribution, assumed to belong to $H_0 \cup H_1$.

Examples of hypothesis testing abound, including, to name a few, goodness-of-fit and testing validity of a purported statistical model, testing stationarity of a time series, hypothesis tests for sequential decision making such as change-point detection, and so on. Central to our study is one of the most fundamental hypotheses testing problems, dating back to Pearson [1900]: that of statistical independence. In contexts where the underlying data (i.e., samples) is personal or conveys sensitive information, guaranteeing privacy (of data samples) in hypothesis testing has recently become the focus of much theoretical and practical interest, both for legal and societal reasons. Indeed, privacy-preserving computation is necessitated by a range of motivations, including regulatory compliance, personal preferences, confidentiality, as well as a migration to distributed forms of computation where client data or intermediate computations of their data are shared among various (not necessarily trusted) parties. In this paper, we quantify privacy in the framework of differential privacy, a rigorous and widely used notion of privacy introduced in Dwork et al. [2014], and focus on a two-party setting. The following is the main problem of consideration in this paper.

Problem statement: How can the sample test-statistic for a hypothesis test of independence between two random variables of arbitrary dimension be estimated privately when samples from these random variables are held by two different parties? The central importance of a test-statistic (to hypothesis testing) is detailed in the preliminaries (Section 2).

Communication setup: In terms of communication, we consider a two-party setup, with Alice holding $X$ and Bob holding $Y$. The communication is one-way, from Alice to Bob: Alice sends an intermediate computation in a privatized form to Bob, who then finishes the rest of the computation on its premise. Bob never reveals the answer to Alice and therefore the on-device computation of Bob involves its own non-privatized data. Therefore, Bob obtains the results (on its side) of the hypothesis test of independence between its data and that of Alice’s, while maintaining the privacy of Alice’s (already privatized) data, regardless of the one-way communications that happen. We refer to this setup as one-way local privacy or one-way local differential privacy, as the privatization mechanism used by Alice ensures differential privacy Dwork et al. [2014]. A more general way to think about
it is as a data summary that when published by Alice, allows any analyst to test independence with another data set they themselves hold. We illustrate this setup in Figure 1.

![Figure 1: Model of privacy considered, where Alice releases a private summary of data as a sufficient statistic which allows any analyst (here, either Bob or Carol) to test independence with another data set they hold with respect to that of Alice’s original data. This private statistic suffices to compute the rest of the test statistic for independence testing on any analyst’s premise. We provide utility bounds on the computation of this test-statistic in Section 4.1.](image)

### 1.1 Related Work

In view of the large amount of literature addressing the problem of independence testing over the years in a variety of statistical settings, we focus here on the closest and most relevant to our work.

**Private independence testing for contingency tables.** This first line of work, which aims to develop differentially private independence tests, itself comes in two distinct flavors: one is the so-called *asymptotic regime* (limiting distribution and properties of the test statistic as sample size goes to infinity), and the second is the *finite-sample regime* (coarser utility guarantees, but with explicit, finite bounds on the sample size required to achieve them).

1. **Asymptotic tests:** The work of Gaboardi et al. [2016] considers independence testing in the (central) model of differentially privacy, from an *asymptotic* perspective: namely, they propose a differentially private analogue of the classical chi-squared tests of independence, and analyse the limiting distribution of the test-statistic along with its resulting power (1-Type II error). Note that, Type II error is the probability of failing to reject null hypothesis when the null hypothesis is not true.

2. **Non-asymptotic tests:** Later work by Sheffet [2018] focuses on the *finite sample* (non-asymptotic) version of the test under a more stringent model of local privacy. The question is formulated, in a manner that is standard in theoretical computer science (specifically in the domain of distribution testing) and in minimax analysis in statistics as part of a composite hypothesis testing problem. They consider a set of product distributions as part of the null hypothesis and frame the alternative hypothesis to contain all the distributions that are “far” from product distributions, where “farness” is quantified by the total variation distance. They focus on the minimax sample complexity achievable using a specific locally private mechanism of Randomized Response [Warner, 1965]. Subsequent work by Acharya et al. [2021] improves on these results, by showing a way to achieve significantly lower sample complexity (still in the locally private setting) by considering a different privacy mechanism and upon establishing matching lower bounds.

**Non-private independence testing with dependency measures.** There have been advances in development of various statistical dependency measures, and an active route of modern independence testing is based on derivation of test-statistics that depend on these measures in addition to on other required terms. We now share some related works that fall in this category. Distance covariance was introduced in [Székely et al., 2007] and can be expressed as a weighted $L_2$ norm between the characteristic function of the joint distribution and the product of the marginal characteristic functions.
This concept has also been studied in high dimensions [Székely and Rizzo, 2013, Yao et al., 2018], and for testing independence of several random vectors [Gao et al., 2021]. In Sejdinovic et al., 2013, tests based on distance covariance were shown to be equivalent to a reproducing kernel Hilbert space (RKHS) test for a specific choice of kernel. RKHS tests have been widely studied in the machine learning community, with a survey of the subject given by Harchaoui et al., 2013 [Harchaoui et al., 2005, Heller et al., 2016] in which the Hilbert-Schmidt independence criterion was proposed. These tests are based on embedding the joint distribution and product of the marginal distributions into a Hilbert space and considering the norm of their difference in this space. One drawback of the kernel paradigm here is the computational complexity, though Jitkrittum et al., 2017, 2016 and Zhang et al., 2018 have recently attempted to address this issue.

Conditional independence tests and causal discovery. A conditional measure of dependence called conditional distance correlation was introduced by Wang et al., 2015. The works in Zhang et al., 2012, 2017, Wang et al., 2020, Shen et al., 2022 performed conditional independence testing and applied them to the problem of causal discovery.

To the best of our knowledge, no other work addresses the question of independence testing under differential privacy (be it local or central) other than the differentially private distribution estimation approach (under total variation distance) Diakonikolas et al., 2015 or two-sample goodness-of-fit Acharya et al., 2018 that can be used to obtain sub-optimal sample complexity guarantees for this problem. We note that this body of work differs from ours, both in the distributed model assumed (the way the data is partitioned across users) and in the guarantee provided (dependency measure used). They also restrict themselves to the discrete setting as opposed to ours. Our method can also be applied to test between samples lying in different dimensions. In particular, most of the works discussed above (except for Gaboardi et al., 2016), follow the norm in distribution testing and focus on very stringent notion of minimax testing under total variation distance, which might be overly conservative in many settings.

1.2 Our contributions

Our contributions are threefold, and can be summarized as follows:

1. We provide a mechanism to privatize the test-statistic required to perform a hypothesis test of independence between \( x \in \mathbb{R}^d \) hosted by Alice and \( y \in \mathbb{R}^m \) hosted by Bob under the regime of one-way local differential privacy.
2. We show one can express our test statistic as a ratio of sums of directional variance queries (of non-private data) with respect to ‘specific’ covariance matrices of sensitive data (for the formal definition of directional variance queries, the reader is referred to Section 2). This reduction enables us to privatize the test-statistic via privatization of these covariances. This brings up the next question of deriving utility guarantees for the privatized test-statistic, leading to our next contribution.
3. We derive both lower and upper bounds on the utility of our private estimator in terms of additive and multiplicative errors. For a chosen \( 0 < \eta < 1 \), to achieve \( (\epsilon, \delta) \)-differential privacy our approach results in a multiplicative error factor of \( 1 + \eta \), and an additive error \( \Delta \) bounded as

\[
\frac{2(1 - \eta)^2}{2(1 + \eta)} \leq \Delta \leq \frac{2\tau}{(1 - \eta)(1 - \eta)s - \tau},
\]

where

\[
\tau := \left( \frac{2048 \ln(2/(m + n)\nu) \ln(2/\delta)}{\eta \epsilon^2} \right) \ln^2 \left( \frac{128 \ln(1/(m + n)\nu)}{\eta^2 \delta} \right),
\]

for some \( s > \frac{\tau}{1 - \eta} \). The additive and multiplicative approximations hold together at the same time with probability at least \( 1 - (m + n)\nu \) for the user’s choice of \( \nu \).

1.3 Our techniques: Privatization Strategy

We now give a high-level summary of our approach along with its organization.

Setup: We let \( X_{n \times d} \) and \( Y_{n \times m} \) denote the sample data matrices held by two clients (whom we refer to by) Alice and Bob, respectively. The number of samples is denoted by \( n \) and each sample
lies correspondingly in \( \mathbb{R}^d \) and \( \mathbb{R}^m \), respectively. Note that, in the rest of the paper we refer to rows \( k, l \) in \( X \) using lower-case representation of \( x_k, x_l \). Similarly, we refer to columns \( k, l \) in \( X \) using upper-case representation of \( X_k, X_l \).

As a starting point for the problem of private independence testing, we first provide the test-statistic (or a matrix square root such that \( L \) form of this kind requires extra work, leading to our main result.

Unfortunately, this first attempt falls short of our goal, as the additive and multiplicative errors in \((1.3)\) do not decompose well to provide a more convenient expression of the form in \((1)\). Obtaining a final form of this kind requires extra work, leading to our main result.

**Definition 1.1** (Directional variance). For an \( n \times d \) matrix \( M \), a directional variance query is specified by a unit-length direction \( \mathbf{x} \), and is given by \( \Phi_M(\mathbf{x}) = \mathbf{x}^\top M^\top M \mathbf{x} \).

**Privatization strategy for \( \hat{\Omega}^2(X, Y) \):** We first express \( \hat{\Omega}^2(X, Y) \) as a function of i) a covariance matrix formed by incidence matrices (or a matrix square root) of a specific weighted graph formed over samples in \( X \) as its vertices (that we refer to as incidence-covariance) and ii) the covariance matrix of \( Y \). We then show that \( \hat{\Omega}^2(X, Y) \) is exactly equal to a sum of directional variances of columns of \( Y \) with respect to the incidence-covariance matrix of \( X \).

**Privatization strategy for \( \hat{S}(X, Y) \):** We are able to express \( \hat{S}(X, Y) \) in a similar form as \( \hat{\Omega}^2(X, Y) \), but with some key differences. Namely, we can write it as a function of the covariance of \( X \) (given by \( XX^\top \)) and a data-independent graph Laplacian matrix \( L^S = nI - ee^\top \). We show that this is exactly equal to the sum of directional variances of the columns of incidence matrix (or a matrix square root such that \( L^S = BB^\top \)) with respect to covariance matrix \( XX^\top \). A key difference with respect to the case of \( \hat{\Omega}^2(X, Y) \) is that, in here, the directional variance is with respect to the covariance matrix of \( X \) while in \( \hat{\Omega}^2(X, Y) \) the directional variance is with respect to the incidence-covariance matrix defined over \( X \).

Therefore, in the notation of directional variances, we show that \( \hat{\Omega}^2(X, Y) = \sum_i \Phi_{B(X)}(y_i) \) where \( y_i \) refers to the \( i \)'th column of \( Y \), and \( \hat{S}(X, Y) = \sum_i \Phi_X(g_i) \) where \( g_i \) refers to the \( i \)'th column of \( G \). A utility bound on computing directional variance queries upon privatization of covariance matrices can be expressed in terms of additive and multiplicative errors. We now define the structure of such approximations as below. In the rest of the paper, for presentation simplicity, we refer to \( \hat{\Omega}^2(X, Y) \) and \( \hat{S}(X, Y) \) by \( \hat{\Omega}^2 \) and \( \hat{S} \), respectively.

**Definition 1.2** (Additive and multiplicative approximation). We say a privacy mechanism that privatizes an estimate \( \hat{A}(x) \) using \( A(x) \), provides an \( (\eta, \tau, \nu) \)-approximation, if the following holds.

\[
\Pr \left[ (1 - \eta) \hat{A}(x) - \tau \leq \hat{A}(x) \leq (1 + \eta) \hat{A}(x) + \tau \right] \geq 1 - \nu.
\]

We use \( \hat{\Omega}^2, \hat{S} \) to denote the privatized versions of \( \hat{\Omega}^2 \) and \( \hat{S} \) respectively that can be obtained upon privatization of the covariance matrix \( XX^\top \) and incidence-covariance \( B(X)B^\top (X) \).

\[
\alpha_\ell \frac{\hat{\Omega}^2}{\hat{S}} + \beta_\ell \leq \frac{\hat{\Omega}^2}{\hat{S}} \leq \alpha_u \frac{\hat{\Omega}^2}{\hat{S}} + \beta_u,
\]

**In section \[4\]**, we first show that a privatization of covariances and incidence-covariances results in separate additive and multiplicative approximations for \( \hat{\Omega}^2 \) and \( \hat{S} \) respectively. A first-pass attempt towards using this result towards our main goal of an approximation of the form in \((1)\), just involves a simple rearrangement of the terms using the individual bounds on \( \hat{\Omega}^2 \) and \( \hat{S} \), leading us to the following result with probability \( \geq 1 - 2\nu \) where we have, \( \frac{(1-\eta)\hat{\Omega}^2-x}{(1+\eta)\hat{S}+\tau} \leq \frac{\hat{\Omega}^2}{\hat{S}} \leq \frac{(1+\eta)\hat{\Omega}^2+x}{(1-\eta)\hat{S}+\tau} \). Unfortunately, this first attempt falls short of our goal, as the additive and multiplicative errors in \((1.3)\) do not decompose well to provide a more convenient expression of the form in \((1)\). Obtaining a final form of this kind requires extra work, leading to our main result.
2 Preliminaries

Preliminaries for independence testing To formalise the problem of hypothesis testing for independence, consider random variables $X$ and $Y$ that have densities $f_X$ on $\mathbb{R}^d$ and $f_Y$ on $\mathbb{R}^m$, respectively, and let $Z = (X, Y)$ have density $f$ on $\mathbb{R}^q$, where $q := d + m$. Given independent and identically distributed copies $Z_1, \ldots, Z_n$ of $Z$, we wish to test the null hypothesis that $X$ and $Y$ are independent, denoted $H_0: X \independent Y$, against the alternative hypothesis that $X$ and $Y$ are not independent, denoted by $H_1: X \not\independent Y$. A distribution $D$ is said to be independent, if it is equal to the product of its marginals.

Test-statistic A predominant methodology for hypothesis testing (in the non-private setting) involves computation of a sample test statistic from the data. The key idea is that the asymptotic distribution of a test statistic (that can be estimated from data samples) is derived (and hence, known) upon conditioning on the null hypothesis being true. Therefore, the observed value of the sample test-statistic (a scalar) is compared with this asymptotic distribution in conjunction with a chosen level of confidence to decide if the observed value falls within the acceptance region or the rejection region. The rejection region refers to the observed value of the sample test statistic being highly unlikely given the asymptotic distribution, if the null hypothesis was supposed to be true.

Preliminaries for differential privacy We now take a quick detour into differential privacy. A widely used mathematical notion of privacy is that of differential privacy [Dwork et al., 2014], which we recall below.

Definition 2.1 ((\(\epsilon, \delta\))-Differential Privacy [Dwork et al., 2014]). A randomized algorithm $A : \mathcal{X} \rightarrow \mathcal{Y}$ is $(\epsilon, \delta)$-differentially private, if, for all neighboring datasets $X, X' \in \mathcal{X}$ and for all (measurable) $M \subseteq \mathcal{Y}$, we have $\Pr[A(X) \in M] \leq e^\epsilon \Pr[A(X') \in M] + \delta$ and two datasets are said to be neighboring if they differ in a single element while $\delta$ is a slack probability of a privacy-failure.

Post-Processing Invariance A major advantage is that differentially private mechanisms are immune to post-processing, meaning that an adversary without any additional knowledge about the dataset $X$ cannot compute a function on the output $A(X)$ to violate the stated privacy guarantees.

Preliminaries for energy statistics We introduce a measure of statistical dependence called distance correlation. This is relevant because the test-statistic that we use for independence testing is a function of distance covariance (an un-normalized version of distance correlation). This is relevant because the test-statistic that we use for independence testing is a function of distance covariance (an un-normalized version of distance correlation).

Population distance covariance For random variables $X \in \mathbb{R}^d$ and $Y \in \mathbb{R}^m$ with finite first moments, the population distance covariance [Szekely et al., 2007] between them is a non-negative number given by

$$\Omega(X, Y) = \int_{\mathbb{R}^{d+m}} |f_{X,Y}(t,s) - f_X(t)f_Y(s)|^\alpha w(t,s)dtds,$$

where $f_{X,Y}$ is the joint characteristic function, and $w(t,s)$ is a weight function defined as $w(t,s) = (C(d,\alpha)C(m,\alpha)|t|^d|s|^m)^{-1}$, with $C(d,\alpha) = \frac{2\pi^{d/2}\Gamma((1-\alpha)/2)}{\pi^{d/2}\Gamma((d+1)/2)}$ for chosen values of $\alpha$ which impacts the choice of the norm. More details about this weight function and the reasoning being its appropriateness for evaluating this integral is given in [Szekely et al., 2007].

For random variables that admit a density, the characteristic function is the Fourier transform of the probability density function. Note that, for complex-valued functions such as the characteristic function used in (2), the norm is dependent on itself and its complex conjugate $\hat{f}$ as $|\hat{f}|^2 = \hat{f}\hat{\hat{f}}$.

Estimators for distance covariance We first define $a_{ij} = \|x_i - x_j\|^2$, $b_{ij} = \|y_i - y_j\|^2$, $a_i = \sum_{l=1}^{n} a_{il}$, $b_i = \sum_{l=1}^{n} b_{il}$, $a = \sum_{k,l=1}^{n} a_{kl}$ and $b = \sum_{k,l=1}^{n} b_{kl}$. We now use these quantities to define an unbiased statistical estimator of distance covariance $\hat{\Omega}^2(X, Y)$ as follows.

$$\hat{\Omega}^2(X, Y) = \frac{1}{n(n-3)} \sum_{i \neq j} a_{ij}b_{ij} - \frac{2}{n(n-2)(n-3)} \sum_{i=1}^{n} a_{i}b_{i} + \frac{ab}{n(n-1)(n-2)(n-3)}$$

A generalization to powers of the norm being other than 2, is referred to in the literature as $\alpha$-distance correlation.
We first express the numerator and denominator of the test-statistic as different functions of directional variance. This is useful because the directional variance in our case is solely a function of a specific covariance (specific to the choice of numerator or denominator) of our sensitive data and the non-sensitive $Y$. Therefore, the directional variances corresponding to numerator and denominator of test-statistic is private upon applying the post-processing property of differential privacy after privatization of that specific covariance. The following two results express the numerator and denominator of statistical dependency denoted by $\hat{\Omega}^2(X,Y)$.

$$\frac{\hat{\Omega}^2(X,Y)}{\hat{\Omega}^2(Y,X)} > \left(\phi^{-1}(1 - \alpha/2)\right)^2,$$

where $\phi(\cdot)$ denotes the standard normal cumulative distribution function, and $\alpha$ denotes the achieved significance level. A test rejecting independence of $X$ and $Y$ when $\sqrt{n\hat{\Omega}^2(X,Y)/\hat{\Omega}^2(Y,X)} \geq \phi^{-1}(1 - \alpha/2)$ is said to have an asymptotic significance level of at most $\alpha$.

4 Privatization of test statistic

We first express the numerator and denominator of the test-statistic as different functions of directional variance. This is useful because the directional variance in our case is solely a function of a specific covariance (specific to the choice of numerator or denominator) of our sensitive data and the non-sensitive $Y$. Therefore, the directional variances corresponding to numerator and denominator of test-statistic is private upon applying the post-processing property of differential privacy after privatization of that specific covariance. The following two results express the numerator and denominator of test-statistic in terms of directional variance.

3 Test statistic for independence testing

The test-statistic we consider is a ratio of sample distance covariance (an un-normalized measure of statistical dependency) denoted by $\hat{\Omega}^2(X,Y)$ and a product of average of distances denoted by $\hat{S}(X,Y)$ defined as $\hat{S}(X,Y) = \frac{1}{n^2} \sum_{k,l=1}^{n} \|x_k - x_l\| \|y_k - y_l\|^2$. In order to define the test statistic, we need to define distance covariance denoted as $\hat{\Omega}^2(X,Y) = \hat{R} + \hat{S} - 2\hat{T}$ where,

$$\hat{R} = \frac{1}{n^2} \sum_{k,l=1}^{n} \|x_k - x_l\|^2 \|y_k - y_l\|^2,$$

$$\hat{S} = \frac{1}{n^2} \sum_{k,l=1}^{n} \|x_k - x_l\| \|y_k - y_l\|^2,$$

$$\hat{T} = \frac{1}{n^2} \sum_{k=1}^{n} \sum_{l=1}^{n} \|x_k - x_l\|^2 \|y_k - y_l\|^2.$$

We use $\Gamma(X,Y,\alpha,n)$ to denote the test statistic that rejects independence when

$$\Gamma(X,Y,\alpha,n) = \frac{n\hat{\Omega}^2(X,Y)}{\hat{S}(X,Y)} > \left(\phi^{-1}(1 - \alpha/2)\right)^2,$$

where $\phi(\cdot)$ denotes the standard normal cumulative distribution function, and $\alpha$ denotes the achieved significance level. A test rejecting independence of $X$ and $Y$ when $\sqrt{n\hat{\Omega}^2(X,Y)/\hat{\Omega}^2(Y,X)} \geq \phi^{-1}(1 - \alpha/2)$ is said to have an asymptotic significance level of at most $\alpha$.
Lemma 4.1. Distance covariance $\hat{\Omega}^2(X, Y)$ can be estimated using the Euclidean distance matrix $E_X$ formed over the rows in $X$, the double-centering matrix $J = I - n^{-1}ee^T$, to form an adjacency matrix given by $W(X) = JE_XJ$ and a corresponding graph Laplacian $L^W(X) = D(W(X)) - W(X)$ where $D(W(X))$ is the degree matrix of $W(X)$ to get

$$\hat{\Omega}^2(X, Y) = \frac{1}{n} \sqrt{2} \text{Tr} \left( Y^T L^W(X)Y \right) = \frac{1}{n} \sqrt{2} \text{Tr} \left( X^T L^W(Y)X \right)$$

Proof. The proof is provided in supplementary material.

Now that distance covariance has been expressed in-terms of a specific graph Laplacian, the following corollary follows to reformulate it as a sum of specific kinds of directional variances that depend on this graph Laplacian.

Corollary 4.1.1. Distance covariance can be expressed as a sum of directional variances as $\hat{\Omega}^2(X, Y) = \sum_i \Phi_{B(X)}(y_i)$ where directional variance was defined above in definition 1.1.

Proof. We expand the distance covariance estimator as follows to get this result.

$$\hat{\Omega}^2(X, Y) = \text{Tr} \left( X^T L^W(X)X \right) = \text{Tr} \left( X X^T L^W(Y) \right) = \sum_i \left( X_i^T L^W(Y)X_i \right)$$

$$= \sum_{i=1}^{d} \text{Tr} \left( X_i X_i^T L^W(Y) \right) = \sum_{i=1}^{m} \text{Tr} \left( Y_i Y_i^T L^W(X) \right) = \sum_{i=1}^{m} \text{Tr} \left( Y_i Y_i^T B(X) B^T(X) \right)$$

$$= \sum_{i=1}^{m} \left( Y_i^T B(X) B^T(X)Y_i \right) = \sum_{i=1}^{m} \hat{\Omega}_i(X, Y) = \sum_{i=1}^{m} \Phi_{B(X)}(y_i)$$

Lemma 4.2. We now express the denominator term of the test-statistic as a specific sum of directional variances as follows. The denominator in the test-statistic,

$$\hat{S}(X, Y) = \frac{1}{n^2} \sum_{k,l=1}^{n} \|x_k - x_l\|^2 \frac{1}{n^2} \sum_{k,l=1}^{n} \|y_k - y_l\|^2,$$

can be expressed as a sum of directional variances as $\hat{S}(X, Y) = \sum_i \phi_X(g_i)$ where $g_i$ is the $i$'th column of a matrix $G$ such that $L^S = GG^T$ for a graph Laplacian matrix $L^S$ given by $L^S = nI - ee^T$. We use the superscript $S$ to distinguish from the Laplacian $L^W$ used in the expression for $\hat{\Omega}^2$.

Proof. The proof is provided in supplementary material.

The privatization strategy therefore is to privatize $B(X) B^T(X)$ in order to privatize $\hat{\Omega}^2(X, Y) = \sum_i \phi_{B(X)}(y_i)$. This privatizes the numerator of the test-statistic. With regards to the denominator, the strategy is to privatize $X X^T$ in order to privatize $\hat{S}(X, Y) = \sum_i \phi_X(g_i)$. Note that in both cases, the privatization required is with respect to covariance matrices. Privatization of covariances was studied in [Blocki et al., 2012, Amin et al., 2019, Biswas et al., 2020]. We summarize our proposed mechanism called $\pi$-test in Figure 2. With covariance matrices being central to many down-stream queries, the question of understanding the effect of their privatization on utility of downstream queries often arises.

4.1 Utility results

In that spirit, the work in [Blocki et al., 2012] provides a utility analysis of directional variances (expressed as functions of covariances) upon privatization of these covariances. This utility bound is formulated in terms of additive and multiplicative errors as part of a probability bound. The notion of additive and multiplicative approximation was defined in preliminaries under definition 1.2. This result only provides us individual bounds (with additive and multiplicative factors) separately for each summand (in summation of directional variances) in the numerator and denominator of our test-statistic. But we need a single bound (with additive and multiplicative factors) for the test-statistic in its entirety, which is a ratio of sums of directional variances. The following results
that we workout in the rest of the paper lets us obtain that bound. We denote each summand in the sum of directional variances corresponding to the numerator and denominator by $\hat{\Omega}_i^2$ for $i \in \{1, 2, \ldots, m\}$ and $\hat{S}_i$ for $i \in \{1, 2, \ldots, k \leq n\}$, respectively, where $k$ denotes the rank of $G$. That is, $\hat{\Omega}^2 = \sum_i \hat{\Omega}_i^2 = \sum_i \phi_{B(X)}(y_i)$ and $\hat{S} = \sum_i \hat{S}_i = \sum_i \phi_{X}(g_i)$.

We use $\hat{\Omega}_i^2, \hat{S}_i$ to denote the privatized versions of $\Omega_i^2$ and $S_i$ respectively that can be obtained upon privatization of the covariance matrix $XX^T$ and incidence-covariance $B(X)B^T(X)$. As a starting point, [Blocki et al., 2012] provided these additive and multiplicative error bounds using $r = \frac{8 \ln(2/\nu)}{\eta}$, $w = \frac{16 \sqrt{r \ln(2/\nu)}}{\ln(16r/\delta)}$ for utility of a single directional variance query expressed using private covariance, which in our setting refers to $\Omega_i^2$ and $S_i$ for any $i \in \{1, 2, \ldots, k \leq n\}$.

$$\mathbb{P}\left((1 - \eta)\hat{\Omega}_i^2 - \tau \leq \hat{\Omega}_i^2 \leq (1 + \eta)\hat{\Omega}_i^2 + \tau\right) \geq 1 - \nu,$$

and similarly for $S$ as

$$\mathbb{P}\left((1 - \eta)\hat{S}_i - \tau \leq \hat{S}_i \leq (1 + \eta)\hat{S}_i + \tau\right) \geq 1 - \nu.$$

But we need a bound on the ratio of specific directional variances. From the inequality on the intersection of $e$ events $E_1, \ldots, E_e$ given as $\mathbb{P}(\cap_{i=1}^e E_i) \geq \sum_{i=1}^e \mathbb{P}(E_i) - (e - 1)$, for $\nu \geq 1/(m + n)$, it follows that

$$\mathbb{P}\left((1 - \eta)\hat{\Omega}^2 - m\tau \leq \hat{\Omega}^2 \leq (1 + \eta)\hat{\Omega}^2 + m\tau\right) \geq 1 - m\nu,$$

and

$$\mathbb{P}\left((1 - \eta)\hat{S} - n\tau \leq \hat{S} \leq (1 + \eta)\hat{S} + n\tau\right) \geq 1 - n\nu.$$

We now provide results for our lower and upper bounds.

**Figure 3:** Privacy-utility trade offs ($\epsilon$ vs mean % relative error & std.dev) for test-statistic estimates with 50 replications using $\eta = 0.05$ (red) & $\eta = 0.1$ (blue).

**Figure 4:** Privacy-utility trade offs ($\epsilon$ vs mean % relative error & std.dev) for $S$ estimates with 50 replications using $\eta = 0.05$ (red) & $\eta = 0.1$. (blue).

**Figure 5:** Privacy-utility trade offs ($\epsilon$ vs mean % relative error & std.dev) for $\Omega^2$ estimates with 50 replications using $\eta = 0.05$ (red) & $\eta = 0.1$.

### 4.1.1 Lower bound

**Theorem 4.3.** For $S > \frac{m\tau}{\nu} \text{ and } n \gg m$, we have the following lower bound on $\frac{\hat{\Omega}^2}{S}$.

$$\mathbb{P}\left(\frac{\hat{\Omega}^2}{S} \geq \frac{1 - \eta}{1 + \eta} \left(\frac{\hat{\Omega}}{S}\right) - \frac{2}{1 - \eta}\right) \geq 1 - (m + n)\nu.$$

**Proof.** The proof is provided in supplementary. \hfill \Box
4.1.2 Upper bound

**Theorem 4.4.** For some $s > \frac{\tau}{1 - \eta}$ and $n \gg m$, we have the following upper bound on $\tilde{\Omega}^2 S$.

$$\mathbb{P} \left( \frac{\tilde{\Omega}^2}{S} \leq 1 + \eta \left( \frac{\tilde{\Omega}^2}{S} \right) + \frac{\tau}{1 - \eta} \left( 1 - \eta \right) s - \tau \right) \geq 1 - (m + n)\nu. \tag{8}$$

**Proof.** The proof is provided in supplementary.

Although, the primary contribution of our paper is theoretical, we perform a brief empirical utility analysis of our mechanism using the publicly available BostonHousing dataset. We share the privacy-utility trade-offs of our private estimator of the test-statistic in Figure [5] for varying values of $\epsilon$, for a $\delta = 0.0002$. We measure utility using relative error (in %), also referred to as percent error for two values of $\eta = 0.05$ and $\eta = 0.1$. We also study this trade-off in individually understanding the utility of the denominator (in Figure [4]) and numerator (in Figure [5]) of the test-statistic.

5 Conclusion

Current works on private independence testing focus on discrete data (contingency tables) and is based on stringent minimax testing under total variation distance, which might be overly conservative in many settings. Our work differs in this sense in terms of distributed model assumed (the way the data is partitioned across users), being compatible with continuous data, its applicability to samples lying in different dimensions unlike these prior methods. While information-theoretic formulations are fairly matured, inference with energy statistics in general brings a newer viewpoint to private testing problems.

6 Limitations and future work

That said, universality results on power analysis of distance covariance based tests are being formulated by various groups in recent preprints such as [Han and Shen 2021]. A further analysis of our test from this viewpoint is needed as part of future work. Although, we used the privacy mechanism for covariance given in [Blocki et al. 2012] as part of our $\pi$-test protocol, other options such as [Biswas et al., 2020, Amin et al., 2019] could be used and theoretical effects of their utility in privatizing the test-statistic is of open interest. We used [Blocki et al., 2012] given it’s suitability to our theoretical study. Similarly, the question of coming up with mechanisms for private conditional independence testing using conditional distance covariance is still of open interest.

A Lower bound

**Theorem.** For $\hat{S} > \frac{\tau r}{1 - \eta}$ and $n \gg m$, we have the following lower bound on $\tilde{\Omega}^2 S$.

$$\mathbb{P} \left( \frac{\tilde{\Omega}^2}{S} \geq 1 + \eta \left( \frac{\tilde{\Omega}^2}{S} \right) - \frac{(1 - \eta)^2}{2(1 + \eta)} \right) \geq 1 - (m + n)\nu. \tag{8}$$

**Proof.** From the naive re-arrangement of the individual bounds, we have the following with probability $\geq 1 - (m + n)\nu$

$$\frac{(1 - \eta)\hat{\Omega}^2 - m\tau}{(1 + \eta)\hat{S} + n\tau} \leq \frac{\Omega^2}{S} \leq \frac{(1 + \eta)\hat{\Omega}^2 + m\tau}{(1 - \eta)\hat{S} - n\tau}. \tag{8}$$

Rearranging the lower bound on $\frac{\Omega^2}{S}$ yields

$$\frac{\tilde{\Omega}^2}{S} \geq \frac{(1 - \eta)\hat{\Omega}^2 - m\tau}{(1 + \eta)\hat{S} + n\tau} = \frac{(1 - \eta)\hat{\Omega}^2 - m\tau}{(1 + \eta)\hat{S} \left( 1 + \frac{n\tau}{(1 + \eta)\hat{S}} \right)} = \frac{(1 - \eta)\hat{\Omega}^2 \left( 1 + \frac{n\tau}{(1 + \eta)\hat{S}} \right) - m\tau - \frac{(1 - \eta)\hat{\Omega}^2 n\tau}{(1 + \eta)\hat{S}}}{(1 + \eta)\hat{S} \left( 1 + \frac{n\tau}{(1 + \eta)\hat{S}} \right)} \tag{9}$$
For $\hat{S} > \frac{n\tau}{1-\eta}$, we have
\[
\frac{m\tau(1 + \eta)\hat{S} + n\tau(1 - \eta)\hat{\Omega}^2}{(1 + \eta)\hat{S}} \leq \frac{m\tau(1 + \eta) + n\tau(1 - \eta)}{(1 + \eta)n} \leq \frac{m(1 + \eta) + n(1 - \eta)}{2n(1 + \eta)}.
\]
where (a) follows from Lemma C.1, i.e., $\hat{\Omega}^2 \leq \hat{S}$, and (b) follows by replacing $\hat{S}$ with $\frac{n\tau}{1-\eta}$ since $\hat{S} > \frac{n\tau}{1-\eta}$. For $n \gg m$, we can approximate (10) as follows
\[
\frac{m\tau(1 + \eta)\hat{S} + n\tau(1 - \eta)\hat{\Omega}^2}{(1 + \eta)\hat{S}} \leq \frac{m(1 + \eta) + n(1 - \eta)}{2n(1 + \eta)} \approx \frac{(1 - \eta)^2}{2(1 + \eta)}.
\]
Plugging (11) into (9) yields
\[
\frac{\hat{\Omega}^2}{S} \geq \left(1 - \eta\right)
\]
\[
\frac{\hat{\Omega}^2}{S} - \frac{(1 - \eta)^2}{2(1 + \eta)}
\]
\[
\tag{12}
\]

B Upper bound

Theorem. For some $s > \frac{\tau}{1 - \eta}$ and $n \gg m$, we have the following upper bound on $\frac{\hat{\Omega}^2}{S}$.
\[
\mathbb{P}\left[\frac{\hat{\Omega}^2}{S} \leq \frac{1 + \eta}{1 - \eta} \left(\frac{\hat{\Omega}^2}{S}\right) + \frac{\tau}{(1 - \eta)s - \tau}\right] \geq 1 - (m + ns)\nu.
\]

Proof. Rearranging the upper bound on $\frac{\hat{\Omega}^2}{S}$, given in (8), yields
\[
\frac{\hat{\Omega}^2}{S} \leq \frac{(1 + \eta)\hat{\Omega}^2 + m\tau}{(1 - \eta)\hat{S}} = \frac{(1 + \eta)\hat{\Omega}^2 (1 - \frac{n\tau}{(1 - \eta)\hat{S}}) + \frac{m\tau(1 + \eta) + n\tau(1 - \eta)}{(1 - \eta)\hat{S}}}{(1 - \frac{n\tau}{(1 - \eta)\hat{S}})}
\]
\[
= \frac{(1 + \eta)\hat{\Omega}^2}{1 - \eta} + \frac{m\tau(1 + \eta)\hat{S} + n\tau(1 + \eta)\hat{\Omega}^2}{(1 - \eta)\hat{S}(1 - \eta)\hat{S} - n\tau}.
\]
\[
\tag{13}
\]
We have
\[
\frac{m\tau(1 - \eta)\hat{S} + n\tau(1 + \eta)\hat{\Omega}^2}{(1 - \eta)\hat{S}} \leq \frac{m\tau(1 - \eta) + n\tau(1 + \eta)}{(1 - \eta)(1 - \eta)\hat{S} - n\tau)} \leq \frac{m\tau(1 - \eta) + n\tau(1 + \eta)}{(1 - \eta)(1 - \eta)ns - n\tau)}.
\]
\[
\tag{14}
\]
where (a) follows from Lemma C.1, i.e., $\hat{\Omega}^2 \leq \hat{S}$, and (b) follows since $\hat{S} \geq ns$, for some $s > \frac{\tau}{1 - \eta}$.
For $n \gg m$, (14) is approximated as
\[
\frac{m\tau(1 - \eta)\hat{S} + n\tau(1 + \eta)\hat{\Omega}^2}{(1 - \eta)\hat{S}} \leq \frac{m\tau(1 - \eta) + n\tau(1 + \eta)}{(1 - \eta)(1 - \eta)ns - n\tau)} \approx \frac{\tau}{(1 - \eta)s - \tau}.
\]
\[
\tag{15}
\]
Plugging (15) into (13) yields
\[
\frac{\hat{\Omega}^2}{S} \leq \frac{(1 + \eta)\hat{\Omega}^2}{(1 - \eta)S} + \frac{\tau}{(1 - \eta)s - \tau}.
\]
\[
\tag{16}
\]
C Error bound on ratio of estimators

Lemma C.1. Assuming that $d_{\text{max}}/d_{\text{min}}^2 \leq \frac{(n-1)}{2}$, where $d_{\text{max}} = \max_{i,j} \|x_i - x_j\|^2$ and $d_{\text{min}} = \min_{i,j,i\neq j} \|x_i - x_j\|^2$, we have $\Omega^2 \leq S$.

Proof. We have

\[
\Omega^2 = \frac{1}{n} \sqrt{2 \text{Tr} \left( Y^T L X Y \right)} = \frac{1}{n} \sqrt{\sum_{i,j} W_{i,j} \|y_i - y_j\|^2}
\]

\[
= \frac{1}{n} \sqrt{\sum_{i,j} |JEX|_{i,j} \|y_i - y_j\|^2}, \quad (17a)
\]

\[
S = \frac{1}{n^2} \sum_{i,j} \|x_i - x_j\|^2 \sum_{i,j} \|y_i - y_j\|^2. \quad (17b)
\]

Assuming that there are a total of $c$ classes in the dataset, we denote by $C_l$ the set of data samples indices that belong to the label $l$, $l = 0, \ldots, c-1$. For $i \in C_l$, we assume that $y_i$ is a vector with only one non-zero entry at the $l$-th coordinate, where for simplicity we set the value in the $l$-th coordinate to 1. Accordingly, for $l = 0, \ldots, c-1$, we have

\[
\|y_i - y_j\|^2 = \begin{cases} 0, & i, j \in C_l \\ 2, & \text{otherwise}, \end{cases} \quad (18)
\]

which results in

\[
\Omega^2 = \frac{1}{n} \sqrt{\frac{2}{n} \sum_{i,j,(i,j) \notin C_l^2} |JEX|_{i,j} \|y_i - y_j\|^2}
\]

\[
\leq \frac{1}{n} \sqrt{\frac{2}{n} \sum_{i,j} |JEX|_{i,j} \|y_i - y_j\|^2}
\]

\[
\overset{(a)}{=} \frac{1}{n} \sqrt{\frac{2}{n} \sum_{i,j} \|x_i - x_j\|^2}, \quad (19)
\]

where (a) follows from Lemma [D.1]. For simplicity of the analysis, we assume that the number of data samples with each class is uniform, i.e., $|C_l| = n/c$, in which case $\sum_{i,j} \|y_i - y_j\|^2 = 2n^2(1 - 1/c)$. Accordingly, we have

\[
S = \frac{2(1 - 1/c)}{n^2} \sum_{i,j} \|x_i - x_j\|^2, \quad (20)
\]

which is minimized for $c = 2$ (assuming that $c \geq 2$), i.e.,

\[
S \geq \frac{1}{n^2} \sum_{i,j} \|x_i - x_j\|^2. \quad (21)
\]

According to (19) and (21), in order to prove $\Omega^2 \leq S$, it suffices to show that

\[
\sum_{i,j} \|x_i - x_j\|^2 \leq \frac{1}{2n} \left( \sum_{i,j} \|x_i - x_j\|^2 \right)^2. \quad (22)
\]

We have

\[
\sum_{i,j} \|x_i - x_j\|^2 \leq n(n-1)d_{\text{max}},
\]

\[
\frac{1}{2n} \left( \sum_{i,j} \|x_i - x_j\|^2 \right)^2 \geq \frac{n^2(n-1)^2}{2n} d_{\text{min}}^2. \quad (23)
\]

Accordingly, having $d_{\text{max}} \leq \frac{(n-1)}{2} d_{\text{min}}^2$ guarantees (22). \qed
D Double-centering lemmas

Lemma D.1. For the centering matrix given by \( J = I - \frac{1}{n}ee^\top \), we have the following property when applied to Euclidean distance matrices of data sample \( X \) denoted by \( E_X \).

\[
\sum_{i,j} [JE_X J]_{i,j} = \frac{1}{n} \sum_{i,j} d_{ij}^2(X)
\]

Proof. We have

\[
JE_X J = \left( I - \frac{1}{n}ee^\top \right) E_X \left( I - \frac{1}{n}ee^\top \right)
\]

(24)

\[
= E_X - \frac{1}{n}ee^\top E_X - \frac{1}{n}E_X ee^\top + \frac{1}{n^2}ee^\top E_X ee^\top,
\]

(25)

according to which

\[
(JE_X J)_{ij} = d_{ij}^2(X) - \frac{1}{n} \sum_{i=1}^n d_{ij}^2(X) - \frac{1}{n} \sum_{j'=1}^n d_{ij'}^2(X) + \frac{1}{n^2} \sum_{i,j'} d_{ij'}^2(X).
\]

(26)

Summing (26) over all \( i, j \) yields

\[
\sum_{i,j} [JE_X J]_{i,j} = \sum_{i,j} d_{ij}^2(X) - \frac{1}{n} \sum_{i,j} \sum_{j'=1}^n d_{ij'}^2(X)
\]

\[- \frac{1}{n} \sum_{i,j} \sum_{j'=1}^n d_{ij'}^2(X) + \frac{n(n-1)}{n^2} \sum_{i,j'} d_{ij'}^2(X)
\]

(27)

\[
= \sum_{i,j} d_{ij}^2(X) - \frac{n-1}{n} \sum_{i,j} d_{ij}^2(X) - \frac{n-1}{n} \sum_{i,j} d_{ij}^2(X) + \frac{n-1}{n} \sum_{i,j} d_{ij}^2(X)
\]

(28)

\[
= \frac{1}{n} \sum_{i,j} d_{ij}^2(X).
\]

(29)

\[
= -\frac{1}{2} J E_X J
\]

(30)

Lemma D.2. The data matrix \( X \) and its Euclidean distance matrix \( E_X \) can be connected using the centering matrix given by \( J = I - \frac{1}{n}ee^\top \) as,

\[
J X^T X J = -\frac{1}{2} J E_X J
\]

Proof.

\[
\left( \|x_i - x_j\|^2 = x_i^T x_i + x_j^T x_j - 2x_i^T x_j \right)
\]

(31)

\[
x_i^T x_j = -\frac{1}{2} \left( \|x_i - x_j\|^2 - \|x_i\|^2 - \|x_j\|^2 \right)
\]

(32)

\[
X^T X = -\frac{1}{2} E_X + \frac{1}{2} \delta 1^T + \frac{1}{2} 1\delta^T \text{ where } \delta \text{ is a vector with } \delta_i = \|x_i\|.
\]

(33)

\[
J X^T X J = -\frac{1}{2} J E_X J + \frac{1}{2} J \delta 1^T J + \frac{1}{2} J 1\delta^T J
\]

But since, \( 1^T J = 0 \) and \( J 1 = 0 \), we have

\[
J X^T X J = -\frac{1}{2} J E_X J
\]

\( \square \)
E Proof of Lemma C.1

We have
\[
\Omega^2(X, Y) = \frac{1}{n} \sqrt{2} \text{Tr} (Y^T L_X Y) = \frac{2}{n^2} \sum_{ij} W_{ij}^X d_{ij}^2(Y)
\]
\[
= \frac{1}{n} \sqrt{\sum_{ij} [J E_X J]_{ij} d_{ij}^2(Y)}.
\]

F Laplacian formulation of \( \Omega^2(X, Y) \) fitness

Lemma F.1. Distance covariance \( \hat{\Omega}^2(X, Y) \) can be estimated using the Euclidean distance matrix \( E_X \) formed over the rows in \( X \), the double-centering matrix \( J = I - n^{-1} e e^T \), to form an adjacency matrix given by \( W(X) = J E_X J \) and a corresponding graph Laplacian \( L^W(X) = D(W(X)) - W(X) \) where \( D(W(X)) \) is the degree matrix of \( W(X) \) to get
\[
\hat{\Omega}^2(X, Y) = \frac{1}{n} \sqrt{2} \text{Tr} (Y^T L_X Y) = \frac{1}{n} \sqrt{2} \text{Tr} (X^T L^W(Y) X)
\]

Proof. This result and corresponding proof is from Vepakomma et al. [2018]. We replicate the same over here for quick reference. Given matrices \( E_X, E_Y, \) and column centered matrices \( X, Y, \) from result of Torgerson [1952] we have that \( \hat{E}_X = -2 X X^T \) and \( \hat{E}_Y = -2 Y Y^T \). In the problem of multidimensional scaling (MDS) (Borg and Groenen, 2005), we know for a given adjacency matrix say \( W \) and a Laplacian matrix \( L \),
\[
\text{Tr}(X^T LX) = \frac{1}{2} \sum_{i,j} [W]_{ij} [E_X]_{i,j}
\]
Now for the Laplacian \( L = L_X \) and adjacency matrix \( W = \hat{E}_Y \) we can represent \( \text{Tr}(X^T L_Y X) \) in terms of \( \hat{E}_Y \) as follows,
\[
\text{Tr}(X^T L_Y X) = \frac{1}{2} \sum_{i,j} \hat{E}_Y_{i,j} [E_X]_{i,j}
\]
From the fact \( [E_X]_{i,j} = (\bar{x}_i, \bar{x}_j) - 2 (\bar{x}_i, \bar{x}_j) \), and also \( \hat{E}_X = -2 X X^T \) we get
\[
\text{Tr}(X^T L_Y X) = -\frac{1}{4} \sum_{i,j=1}^{n} \hat{E}_Y_{i,j} \left( [E_X]_{i,i} + [E_X]_{j,j} - 2 [E_X]_{i,j} \right)
\]
\[
= \frac{1}{2} \sum_{i,j} \hat{E}_X_{i,j} \hat{E}_Y_{i,j} - \frac{1}{4} \sum_{i,j} [E_X]_{i,j} \sum_{j} [E_Y]_{i,j}
\]
\[
- \frac{1}{4} \sum_{i} [E_X]_{i,i} \sum_{j} [E_Y]_{i,j}
\]
Since \( \hat{E}_X \) and \( \hat{E}_Y \) are double centered matrices \( \sum_{i=1}^{n} [\hat{E}_Y]_{i,j} = \sum_{j=1}^{n} [\hat{E}_Y]_{i,j} = 0 \) it follows that
\[
\text{Tr}(X^T L_Y X) = \frac{1}{2} \sum_{i,j} \hat{E}_X_{i,j} \hat{E}_Y_{i,j}
\]
It also follows that
\[
\nu^2(X, Y) = \frac{1}{n^2} \sum_{i,j=1}^{n} [\hat{E}_Y]_{i,j} [E_X]_{i,j} = \frac{2}{n^2} \text{Tr}(X^T L_Y X)
\]
Similarly, we can express the sample distance covariance using Laplacians $L_X$ and $L_Y$ as

$$\hat{\nu}^2(X, Y) = \left( \frac{2}{n^2} \right) \text{Tr} \left( X^T L_Y X \right) = \left( \frac{2}{n^2} \right) \text{Tr} \left( Y^T L_X Y \right)$$

The sample distance variances can be expressed as $\hat{\nu}^2(X, X) = \left( \frac{2}{n} \right) \text{Tr} \left( X^T L_X X \right)$ and $\hat{\nu}^2(Y, Y) = \left( \frac{2}{n^2} \right) \text{Tr} \left( Y^T L_Y Y \right)$ substituting back into expression of sample

\[ \square \]

G  \( S(X, Y) \) as a sum of directional variances

Lemma G.1. We now express the denominator term of the test-statistic as a specific sum of directional variances as follows. The denominator in the test-statistic,

$$\hat{S}(X, Y) = \frac{1}{n^2} \sum_{k,l=1}^n \|x_k - x_l\|^2 \frac{1}{n^2} \sum_{k,l=1}^n \|y_k - y_l\|^2,$$

can be expressed as a sum of directional variances as $\hat{S}(X, Y) = \sum_i \phi_X(g_i)$ where $g_i$ is the $i$'th column of a matrix $G$ such that $L^S = GG^T$ for a graph Laplacian matrix $L^S$ given by $L^S = nI - ee^T$. We use the superscript $S$ to distinguish from the Laplacian $L^W$ used in the expression for $\Omega^2$.

Proof. We first start simple, by expressing the Euclidean distance between any two pairs of points using a matrix trace formulation as follows.

$$d_{ij}^2(X) = \sum_{a=1}^m x_a^T (e_i - e_j)(e_i - e_j)^T x_a = \sum_{a=1}^m x_a^T A_{ij} x_a = \text{tr} X^T A_{ij} X$$

where $A_{ij} = (e_i - e_j)(e_i - e_j)^T$. Similarly, $\sum_{i<j} d_{ij}^2(X) = \text{tr} X^T \left( \sum_{i<j} d_{ij}^{-1}(X) A_{ij} \right) X$. We now extend this to express the sum of all pairs of Euclidean distance matrices as follows.

$$\eta(X) = \sum_{i<j} w_{ij} d_{ij}^2(X) = \text{tr} X^T \left( \sum_{i<j} w_{ij} A_{ij} \right) X = \text{tr} X^T L^S X$$

where $L^S = \left( \sum_{i<j} w_{ij} A_{ij} \right)$. As we would like to express the sum of all pairs of distances, we consider the case where $W_{ij} = 1, \forall i, j \in [n]$. Note that this matrix exactly has the structure of a graph Laplacian. This results in a graph Laplacian $L^S$ where

$$L^S = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & \ddots & \ddots & \cdots \\ \cdots & \cdots & -1 & n-1 \end{bmatrix} = nI - ee^T$$

Therefore,

$$S^S(X, Y) = \eta(X) \eta(Y) = \frac{4}{n^2} \text{Tr} \left( X^T L^S X \right) \cdot \text{Tr} \left( Y^T L^S Y \right)$$

Note that we could also express $\text{Tr} \left( X^T L^S X \right)$ as

$$\text{Tr} \left( X^T L^S X \right) = \text{Tr} \left( X^T GG^T X \right) = \text{Tr} \left( G^T X X^T G \right) = \sum_i \left( g_i^T X X^T g_i \right) = \sum_i \phi_X(g_i),$$

where $k$ is the rank of the matrix $L^S$. \[ \square \]
H Population distance correlation

Using the definition of distance covariance, we have the following expression for the square of distance correlation [Szekely et al. [2007]] between random variables

\[
\text{DCOR}(X, Y) = \frac{\Omega^2(X, Y)}{\sqrt{\Omega^2(X, X) \Omega^2(Y, Y)}},
\]

if \( \Omega^2(X, X) \Omega^2(Y, Y) > 0 \)

\[
0,
\]

if \( \Omega^2(X, X) \Omega^2(Y, Y) = 0 \)

This always lies within the interval \([0, 1]\) with 0 referring to independence and 1 referring to dependence.

I Families of some dependency measures

1. **Energy statistics**: Distance correlation (DCOR), Brownian distance covariance, Partial distance correlation, Partial martingale difference correlation
2. **Kernel covariance operators**: Constrained covariance (COCO), Hilbert-Schmidt independence criterion (HSIC), Kernel Target Alignment (KTA)
3. **Integral probability metrics**: Maximum mean discrepancy (MMD), Wasserstein distance, Dudley metric and Fortet-Mourier metric, Total variation distance.
4. **Information theoretic measures**: Mutual information, f-divergence, Renyi divergence, Hellinger distance, Total variation distance, Maximal information coefficient, Total information coefficient

References


Karl Pearson. On the criterion that a given system of deviations from the probable in the case of a correlated system of variables is such that it can be reasonably supposed to have arisen from random sampling. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 50(302):157–175, 1900.


