Splintering with distributions:
A stochastic decoy scheme for private computation

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Abstract
Performing computations while maintaining privacy is an important problem in today's distributed machine learning solutions. Consider the following two setups between a client and a server, where in setup i) the client has a public data vector $x$, the server has a large private database of data vectors $B$ and the client wants to find the inner products $\langle x, y_k \rangle, \forall y_k \in B$. The client does not want the server to learn $x$ while the server does not want the client to learn the records in its database. This is in contrast to another setup ii) where the client would like to perform an operation solely on its data, such as computation of a matrix inverse on its data matrix $M$, but would like to use the superior computing ability of the server to do so without having to leak $M$ to the server.

We present a stochastic scheme for splitting the client data into privatized shares called splinters. This enables client devices to receive services from server without directly sharing any sensitive data. The server performs computations over these splinters and the corresponding results are sent back to the client. The client has required private coefficients to perform specific operations referred to as unsplintering over these intermediate results in order to obtain the required final result as if the sensitive data itself was sent across to the server instead of the communicated splinters. The approach is relatively communication efficient, supports composition of operations and does not require a trusted non-colluding third party to mediate the process. Replication/download of the sensitive database is not required during this process. We show an

1. Introduction
We describe a method for private computation in distributed machine learning via privatized shares of sensitive data called splinters. This enables client devices to receive services from server without directly sharing any sensitive data. The server performs computations over these splinters and the corresponding results are sent back to the client. The client has required private coefficients to perform specific operations referred to as unsplintering over these intermediate results in order to obtain the required final result as if the sensitive data itself was sent across to the server instead of the communicated splinters. The approach is relatively communication efficient, supports composition of operations and does not require a trusted non-colluding third party to mediate the process. Replication/download of the sensitive database is not required during this process. We show an
application of this proposed method for privately performing operations such as inner product and matrix inverse. Such a private computation with such fundamental operations results in a wide array of societal applications such as private machine learning, private information retrieval, private set intersection, private stream searching, private scientific computation and private information storage.

1.1 Contribution

The contributions in our paper can be enumerated as follows.

1. We propose a method called Splintering for private computation between two entities. The method involves stochastically splitting the private data vector on the client into publicly shareable privacy preserving shares called splinters and private coefficients. The server performs specific computations on these splinters and returns the obtained intermediate results to the client. The client further performs some computation on these intermediate results to obtain the desired final result.

2. We share examples of our scheme for private computation of fundamental operations such as that of a matrix inverse, inner product or for ascertaining set inclusion or to find k-nearest neighbors given a private query vector on a client device.

3. We enhance the security of our method by pairing each splinter with many decoy splinters that cannot be distinguished from the original splinter. We back this up with theory as well as with experimental results using state of the art methods like XGBoost that fail to classify the decoy splinters from the original splinter. We show that these classification models end up performing only up to a chance accuracy, even after rigorous tuning. This protects the client data from being reconstructed by a malicious server as we theoretically show that the number of samples the server has access to is much below the sample complexity required to perform a decent statistical recovery that leaks the raw data of the client. We also share theoretical results that prevent reconstruction of servers data by a malicious client.

4. To learn decoy distributions that can generate the decoy splinters with security guarantees, we connect distance correlation, a statistical measure of dependency whose sample statistic is based on distances to Kullback-Leibler divergence between Multivariate Gaussians. Population distance correlations is dependent on a distance between the joint characteristic functions (Fourier transform of probability distribution) and marginal characteristic functions. We show an elegant separability property that allows for efficient estimation of Gaussian mixture distributions that optimize Kullback-Leibler divergence with respect to a target Gaussian mixture. We use these learnt distributions to generate our decoy splinters with theoretical guarantees that an estimation of their population parameters requires a high sample complexity, thereby protecting the client data.

1.2 Organization of this paper

We re-introduce splintering as part of the related work section along with other related works. In section 2, we propose examples of splintering to perform operations such as matrix inverse. In section 3, we introduce a naive way to generate splinters along with hardness of recovery guarantees for privacy. In section 4, we introduce a better way to generate splinters using Gaussian mixtures and share hardness of recovery guarantees. In section 5, we introduce a notion of decoy splinters to further secure this method along with directly useful theoretical results from an information theoretic perspective in order to generate these decoy splinters. In section 6, we share some experimental
results to empirically back the theory in our scheme from section 5. We conclude the paper with an appendix followed by a bibliography.

1.3 Related work

We share relevant related work to our proposed method and categorize them accordingly as follows:

1. **Splintering for private querying** Our work is an extension of our earlier work on Splintering in (Vepakomma* and Raskar*, 2019; Raskar et al., 2019). We share more details about splintering in the next section, given its direct relevance to the context of this paper.

2. **Private information retrieval**
   PIR schemes introduced by Chor et al. (1995) look at the problem of devising a communication protocol involving just two parties, the database and the user, each having a secret input of data records $r_1, r_2, \ldots, r_n$ and a query record $q$ respectively such that the user can retrieve a record $r_i$ from the database, while keeping $i$ private from this database entity. The PIR schemes can be broadly categorized into information theoretic PIR (Chor and Gilboa, 1997; Kushilevitz and Ostrovsky, 1997b; Cachin et al., 1999). Information theoretic PIR schemes require replication of data across multiple servers. Communication complexity of information theoretic PIR with 2 servers is $O(\sqrt{n})$, for k-servers is $O(n^{1/Ω(k)})$ and for n-servers is $\text{polylog}(n)$ (Chor et al., 1995; Kushilevitz and Ostrovsky, 1997a; Vengroff and Scott Vitter, 1995; Williams and Sion, 2008; Mittal et al., 2011; Melchor, 2008; Goldberg, 2007; Aguilar-Melchor and Gaborit, 2007; Song et al., 2000).

3. **Private set intersection**
   This paradigm aims to privately find an intersection between two sets. The works in private set intersection (PSI) can be classified based on techniques based on public-key coefficient cryptography (Meadows, 1986; De Cristofaro and Tsudik, 2010), generic secure computation (Huang et al., 2012) based on arithmetic and boolean circuits and those based on oblivious transfer (Dong et al., 2013; Pinkas et al., 2014; Freedman et al., 2004; Dachman-Soled et al., 2009; Kerschbaum, 2012; Rindal and Rosulek, 2017; Kamara et al., 2014; Debnath and Dutta, 2015). The initial works based on oblivious transfer assume semi-honest (passive) adversaries. These techniques in conjunction with permutation hashing schemes such as cuckoo hashing (Arbitman et al., 2010) help provide greater scalability in terms of computation (Pinkas et al., 2015; De Cristofaro and Tsudik, 2010; Kiss et al., 2017).

4. **Differentially private SQL queries**
   These techniques for differential private answering of SQL queries allows for SQL predicates and clauses like WHERE, COUNT, MIN, MAX, Top K, HAVING and GROUP BY as shown in (Kotsogiannis et al., 2019a; Wang et al., 2017; Bater et al., 2017; Kotsogiannis et al., 2019b; Suresh et al., 2019). The work in Groce et al., 2019 combines private set intersection with differential privacy. A more relevant version of work for differentially private JOIN queries was presented in (Narayan and Haeberlen, 2012; Chen and Zhou, 2013).

5. **Private stream searching**
   These techniques work on privately matching a query with respect to a data stream. The work in Bethencourt et al., 2009 requires a communication and storage complexity of $O(m \log(t/m))$ where $m$ is upper bound on size of documents and $t$ is the number of documents in the stream.

6. **Private information storage**
   These techniques give protocols for both private reading as well as writing of a database.
(Clarke et al., 2001; 2002; Tajeddine et al., 2018; Ostrovsky and Shoup, 1997; Iliev and Smith, 2004; Kumar et al., 2017) and therefore are suitable for distributed storage systems.

7. **Formal notions of privacy** There are several formally established notions of privacy such as $(\epsilon, \delta)$-differential privacy (Dwork, 2008; Dwork and Smith, 2010; Dwork et al., 2011, 2014; Abadi et al., 2016; Machanavajjhala et al., 2008; Nissim et al., 2012; McSherry and Talwar, 2007; Nissim et al., 2007, 2012, 2017), Lipschitz privacy (Koufogiannis et al., 2015a,b; Chatzikokolakis et al., 2013; Koufogiannis, 2017; Koufogiannis and Pappas, 2016), Blowfish privacy (He et al., 2014; Nie et al., 2010; Machanavajjhala and Kifer, 2015), Pufferfish privacy (Kifer and Machanavajjhala, 2014; Song et al., 2017; Kifer and Machanavajjhala, 2012) which allows the user to specify a class of protected predicates that must be learned subject to the guarantees of differential privacy, and all other predicates can be learned without differential privacy, Minimax filter, ON-OFF privacy, Secure Shuffling, concentrated differential privacy, local differential privacy, central differential privacy and Renyi differential privacy (Mironov, 2017; Wang et al., 2018; Geumlek et al., 2017). Each of these notions of privacy has a formal mathematical definition of privacy. There is a lengthy body of work of several privacy-preserving mechanisms that can help attain one or more of these stringent notions of privacy, depending on the statistical query or statistical model being privatized in the presence of any constraints that might be present.

8. **Cryptographic computation** This can be categorized into techniques for homomorphic encryption (Gentry and Boneh, 2009; Naehrig et al., 2011; Van Dijk et al., 2010; Brakerski and Vaikuntanathan, 2011; Stehle and Steinfeld, 2010; Brakerski and Vaikuntanathan, 2014; Brakerski et al., 2014; Gentry and Halevi, 2011; Fan and Vercauteren, 2012; Smart and Vercauteren, 2010; Damgard et al., 2012; Fontaine and Galand, 2007; Cramer et al., 2001; Sathya et al., 2018) and secure multi-party computation Yao (1982a,b); Ishai et al. (2007); Evans et al. (2017, 2018); Bogetoft et al. (2009); Lindell (2005); Goldreich (1998); Canetti et al. (1996); Cramer et al. (2015); Ben-David et al. (2008); Garg et al. (2014); Hirt et al. (2000); Atallah and Du (2001).

9. **Private data release methods** The work in Chanyaswad et al. (2019) gives a good survey of works on non-interactive private data release. In the case of supervised learning, the private data release methods are categorized into private dimensionality reduction such as in (Jiang et al., 2013; Blocki et al., 2012) and private generative model based approaches such as (Ullman and Vadhan, 2010; Dwork et al., 2009; Bindschaedler et al., 2017). There have been works focusing on other specific queries outside the realm of supervised learning such as for releasing k-way contingency tables in Hardt et al. (2012) via private data release. The work in Gupta et al. (2012) focuses on private data release for the query of releasing graph cuts and Balog et al. (2017) shows a method for private data release based on kernel mean embeddings.

2. **Background on Splintering**

We now detail the first-order idea of splintering introduced in (Vepakomma* and Raskar*) 2019; Raskar et al., 2019 given its direct relevance to this paper.
2.1 Splintering

For a $d$ dimensional input query vector $x$, the client device creates $d$ shares corresponding to $x$ as \{\(z_1, z_2 \ldots z_d\) so that

\[ x = \text{Splint}(z_1, z_2 \ldots z_d), \forall i \in 1..d \]

The most basic splinter function that allows for such a representation is a linear combination using coefficients $\alpha_i$ as

\[ x = \sum_{i=1}^{d} \alpha_i z_i, \forall i \in 1..d \]

The $\alpha'_i$s are private to the client and not shared with any other entity, be it another client or a server. The splinters $z_i$ are shared with the server. The server performs a set of application dependent operations on the splinters $z_i, \forall i \in 1..d$ and sends results $\{\beta_i\}$ back to client on either all or a subset of the $d$ shares. The client performs a local computation called UnSplint using original shares $z_i$, its corresponding $\alpha_i$’s that are known only to the client and received $\beta'_i$s obtained from the server. This unsplinting operation reveals the true result $l$ of the intended application to the client.

\[ l = \text{UnSplint}(\alpha_i, z_i, \beta_i), \forall i \in 1..d \]

Note that although $x$ is represented via a linear combination, the computation of $\{\beta_i\}$ and UnSplint is not necessarily linear.

2.2 Previous examples of Splintering

2.3 k-nearest neighbors

A splintering scheme for the problem of finding the $k$-nearest vectors in a private database on a server with respect to a private query vector on the client was proposed in Vepakomma* and Raskar* (2019).

2.4 Set inclusion

Ascertaining set inclusion and set intersection is an important primitive used in databases, information retrieval and cryptography. Raskar et al. (2019) introduced a splinter based bucketization scheme for the problem of ascertaining set inclusion. The setting that was considered involves the client that attempts to ascertain if its query data vector exists in a database on the server.

2.5 Newer proposals for splintering

We now propose splintering schemes for two more important operations of inner product and matrix inverse.

2.6 Inner product

We propose a splintering scheme for private computation of an inner product in the following setting. The client has a private data vector $x$, the server has a large private database of data vectors $B$ and the client wants to find the inner products $\langle x, y_k \rangle, \forall y_k \in B$. The client does not want the server to learn $x$ while the server does not want the client to learn the records in its database.
2.6.1 Non-malicious client

We first consider the setting where the client is non-malicious and genuinely would like obtain the list of inner products with respect to the servers data while protecting the privacy of its data. In this setting, the client for represents its data record using \(d\) splinters as

\[
x = \sum_{i}^{d} \alpha_i z_i, \forall i \in 1..d
\]

The \(\alpha_i\)'s are private to the client and not shared with any other entity, be it another client or a server. The splinters \(z_i\) are shared with the server. The scheme for generation of splinters is described in sections 3, 4 and 5 along with hardness of recovery guarantees for the clients privacy.

The server returns the inner products of \(\langle z_i, y_k \rangle \forall z_i \in \{z_1, z_2 \ldots z_d\}\) and \(\forall y_k \in B\). The client obtains the inner product for any \(x, y_k\) as

\[
\sum_{i}^{d} \alpha_i \langle z_i, y_k \rangle
\]

The client uses a different set of splinters for every single query with the server.

2.6.2 Malicious client

In the case where the client is malicious and intends to recover all the vectors in the private database, the setting is equivalent to the well known problem of learning mixtures of linear regressions (Mazumdar and Pal, 2020; Yin et al., 2018; Chen et al., 2020; Li and Liang, 2018) which is a generalization of the compressed sensing problem when the cardinality, \(|B| > 1\) as in any typical database.

2.6.3 Statement of learning mixture of sparse linear regressions problem

Given \(L = |B|\) unknown distinct vectors \(y_1, y_2, \ldots, y_L \in \mathbb{R}^p\) and each is \(t\)-sparse meaning that the number of non-zero entries in each \(y_i\) is at most \(t\) where it is a known parameter. We define an oracle on the server which, when queried with a splinter \(\hat{z}_i\), returns the noisy output \(\langle \hat{z}_i, y_k \rangle + \eta\) where \(\eta\) is a random variable with an expectation of 0 and \(y_k\) is uniformly chosen from the \(B\). The goal is to recover all vectors in \(B\) (i.e, data of the server) by making a set of queries \(\hat{z}_1, \hat{z}_2, ..., \hat{z}_m\) to the server.

2.7 Attack scheme for inner product and hardness of recovery of servers database

Therefore the malicious client can recover the servers dataset from the resulting inner products by sending rows of a Vandermonde matrix to the server as its splinters as described in [Krishnamurthy et al., 2019]. The recovery guarantees provided there are as follows. The malicious client can estimate the servers data with \(O(t \log^3 n \exp(\sigma/\epsilon)^2/3)\) number of vandemonde samples where \(t\) is the precision(in terms of number of digits of servers data vector) and \(\sigma\) is the variance of \(\eta\). If there is no noise, then the malicious client can recover all of servers data with \(2tL \log(tL)\) and the recovery is guaranteed with at least \(1 - (3/t)\) probability.

**Condition without any sparsity requirements** Therefore under no sparsity condition, we can change \(k\) to \(p\) in that bound. Where \(p\) is the size of entries in each data record on the server. The bounds will still hold and reveal under no additional noise condition with \(O(2pL \log(pL))\).
Vandermonde samples. So adding noise and changing precision is important to have the other bound which is exponential instead at $O(n \log^3 n \exp(\sigma/\epsilon)^2/3)$ under the no sparsity case while limiting the number of queries from the client according to this complexity, in order for the server to have a protection from the malicious client.

### 2.8 Matrix Inverse

We now propose a splintering scheme for the important operation of matrix inversion in this subsection. We consider a setting where the client has a large sensitive matrix $M_{n \times n}$ and would like to use the service of a computationally powerful server in order to privately obtain the inverse $M^{-1}$. In order to obtain the inverse of a private matrix $M_{n \times n}$, we split it into the form using $A_{n \times n}, U_{n \times k}, V_{k \times n}$ and $Z_2$ of dimension $k \times k$ as

$$M^{-1} = (A + UZ_2V)^{-1}$$

where $A$ is written in terms of a splinter matrix $Z_1$ of dimension $n \times n$ as

$$A = \alpha_1 Z_1$$

and $UZ_2V$ is of the form

$$\begin{bmatrix} \alpha_3 \\ \alpha_3 \\ \alpha_3 \\ \end{bmatrix} U \begin{bmatrix} \alpha_4 \\ \alpha_4 \\ \end{bmatrix}$$

where $Z_2$ is the other splinter. Now by the popular matrix inversion lemma (Sherman-Morrison-Woodbury formula)

$$(A + UZ_2V)^{-1} = A^{-1} - A^{-1} U (Z_2^{-1} + VA^{-1}U)^{-1} VA^{-1}$$

Therefore the proposed scheme now is to send $Z_1, Z_2$ to the server which sends back $Z_1^{-1}, Z_2^{-1}$ to the client that holds $M$ along with the secret coefficients $\alpha_1, \alpha_2, \alpha_3$. The client then obtains the final solution $M^{-1}$ by computing

$$(\alpha_1 Z_1 + UZ_2 V)^{-1} = 1/\alpha_1 Z_1^{-1} - 1/\alpha_1 Z_1^{-1} U (Z_2^{-1} + VA^{-1}U)^{-1} 1/\alpha_1 VZ_1^{-1}$$

#### 2.8.1 Computational Savings

In cases where $n \gg k$, the matrix $(Z_2^{-1} + VA^{-1}U)^{-1}$ which is of dimension $k \times k$ is much easier to invert than the original private data matrix $M_{n \times n}$ thereby offloading the heavier computation onto the server while preserving privacy and requiring a much smaller computation on the client.

### 3. Generation of Splinters

In this section, we first introduce our initial solution for generating the splinters. An improved solution based on Gaussian mixtures is provided in section 4 and this is further improved with decoy splinters in section 5. We back our scheme with hardness of recovery guarantees based on inability to distinguish the distribution of one splinter from its corresponding decoy splinters based on minimum required sample complexity that is in turn influenced by the condition number of estimation and total.
variation distance between the Gaussian mixtures. We therefore learn distributions that satisfy these required conditions in order to be used as generative distributions to sample the corresponding decoy splinters.

### 3.1 Initial solution

We now share details of the initial solution for generating splinters $Z_i$ within the above framework. In this version, $d - 1$ out of $d$ splinters used are data-independent and sampled from a different chosen distribution each with large variance-covariances. The client first generates $d - 1$ data-independent samples as $Z_i \sim \mathcal{N}(0, \Sigma_i), \forall i \in \{1 \ldots d - 1\}$. Only one splinter is dependent on data $X$ and is generated as

$$Z_d = \frac{1}{\alpha_d} (X - \sum_{i \neq d} Z_i)$$

where $\alpha_d$ is the corresponding secret coefficient for the data dependent splinter. The rest of the coefficients are also secret and only known to the client and never shared with the server.

**Rescaling step** Once the data dependent coefficient and splinter have been generated, the rest of the data-independent splinters are scaled by their corresponding secret coefficients as $Z_i = \frac{1}{\alpha_i} Z_i$. All the secret coefficients $\alpha_i$ have are chosen from a $p$-bit base-2 floating point system allowed by the computer architecture, where $p \in \{16, 32, 64\}$. Therefore, in order to reconstruct a data matrix $X$ from scaled $Z_i$’s; one would need access to the secret coefficients. Every communication from the client to server in this scheme involves a different set of splinters.

### 4. MixtureSplinters for improved splinter generation

We now improve upon the initial solution from previous section for this problem of splinter generation. We refer to this improved version as **MixtureSplinters**. We further build upon this version in the next section and proposed a more secure version of this scheme called **DecoySplinters** by introducing a notion of statistical decoys.

In MixtureSplinters, each splinter is generated from a different Gaussian mixture with $m$ components as,

$$z_i \sim \sum_{a=1}^{m} \omega^a_i g^a = \sum_{a=1}^{m} \omega^a_i N \left( x; \mu^a_i, \Sigma^a_i \right)$$

(4)

The client data record $x$ is represented using these splinters, just like in the previous sections as

$$x = \sum_{i} \alpha_i z_i, \forall i \in 1..d$$

### 4.1 Hardness of recovery guarantees for client data

Estimation of Gaussian mixtures has been extensively studied in [Kalai et al., 2012, Kannan et al., 2005, Moitra and Valiant, 2010, Dasgupta, 1999, Ge et al., 2015, Sanjeev and Kannan, 2001, Daskalakis and Kamath, 2014]. Sample complexity for an $\epsilon$-approximation of a $p$-dimensional
Gaussian mixture with \( m \) components is
\[
n > \left( \frac{\kappa(z_i).p}{c.\delta} \right)^{c_m}
\]
samples with probability greater than \( 1 - \delta \) where
\[
\kappa(z_i) = \frac{1}{\min\{\{w_1, w_2, \ldots, w_k\} \cup \{D_{tv}(F_i, F_j)|i \neq j\}\}}
\]
is the condition number of the estimation of the Gaussian mixture.

Therefore, when each \( z_i \) is sampled with a different Gaussian mixture with many components \( m \) such that \( w_i \) is a very small probability or if the total variation distance between individual multivariate Gaussian components, \( F_i, F_j \) is small, it would lead to a very high sample complexity required for any reasonably accurate estimation of the splinter distributions.

5. Generating decoy distributions

We now propose a novel approach of DecoySplinter for further increasing the security of the MixtureSplinter scheme. The idea of a decoy splinter is to generate random splinters that are not supposed to be used in order to obtain a proper reconstruction of the dataset. That is any random splinter that is sent to the server in addition to the d-splinters that form the relation
\[
x = \sum_{i=1}^{d} \alpha_i z_i, \forall i \in 1, 2, \ldots, d
\]
is a decoy splinter. A malicious server would need to set the coefficients of these splinters to 0, in addition to guessing the right secret coefficients of the non-decoy splinters.

5.1 Bruteforce attack success probability

The bruteforce attack probability of success for guessing the right combination of secret coefficients when there are \( d \) non-decoy splinters, \( q - d \) decoy splinters, and each coefficient is a \( b \)-bit number is based on a power tower function in the denominator as
\[
1 \frac{1}{2^{bd} \times \binom{q}{q-d}}
\]
and therefore is infinitesimally small.

5.2 Key insight to motivate the notion of a statistical decoy

For a decoy splinter to be effective, it needs to be such that it is very hard to statistically distinguish between a decoy splinter and a non-decoy splinter. Given the known probability distribution of \( z_i \sim \mathcal{G} \) for a Gaussian mixture, if a decoy Gaussian mixture distribution to sample decoys from as \( d_i \sim \mathcal{H} \) is generated, such that the total variation distance \( D_{tv}(\mathcal{G}, \mathcal{H}) \leq \tau \), then it would require an order of at least \( 1/\tau \) samples to have a constant probability of distinguishing between the case that all samples arise from \( \mathcal{G} \) or all samples arise from \( \mathcal{H} \). The decoy splinter distributions corresponding to
every non-decoy splinter distribution \( g^i \) can be therefore generated such that \( D_{TV}(G, H) \leq \tau \). Note that the covariance and mean of the corresponding Gaussian mixture of each splinter distribution is,

\[
\Sigma_G^i = \sum_a \Sigma_a^i w_a^i + \sum_a (\mu_a^i)^T \mu_a^i w_a^i - (\mu_a^i)^T \mu_a^i
\]

and

\[
\mu_G^i = \sum_a \mu_a^i
\]

By the Pinsker’s inequality we have

\[
D_{TV}(G, H) \leq \sqrt{2D_{KL}(G, H)}
\]

We therefore now show a way to learn a Gaussian mixture from which a decoy shadow can be sampled. We do this by ensuring that the learnt Gaussian mixture of the decoy has a KL-divergence less than \( \tau \) with respect to the high-condition number Gaussian mixture from which the corresponding real shadow was sampled. In addition to these splinters which are tagged correspondingly to non-decoy splinters, we also generate a data independent untagged decoy splinter using the same mixture distribution used to generate the non-decoy splinter. The decoys, untagged decoys and real splinters are all shuffled to not be in an order before sharing with a server.

5.3 Optimizing Average of Bounds on KL-Divergence between two GMMs through distance covariance

For the distribution learning problem motivated in the previous section, the key is to be able to learn a \( \tau \)-close Gaussian mixture to a given target Gaussian mixture. We therefore share some results on KL-divergences between Gaussian mixtures Durrieu et al. (2012). Let \( f \) and \( g \) be two PDFs in \( \mathbb{R}^d \), where \( d \) is the dimension of the observed vectors \( x \). The KL-divergence between \( f \) and \( g \) is defined as:

\[
D_{KL}(f||g) = \int_{\mathbb{R}^d} f(x) \log \frac{f(x)}{g(x)} \, dx
\]

When \( f \) and \( g \) are PDFS of multivariate normals:

\[
D_{KL}(f||g) = \frac{1}{2} \log \frac{\lvert \Sigma_g \rvert}{\lvert \Sigma_f \rvert} + \frac{1}{2} \text{Tr}((\Sigma_g)^{-1} \Sigma_f) + \frac{1}{2} (\mu_f - \mu_g)^T (\Sigma_g)^{-1} (\mu_f - \mu_g) - \frac{d}{2}
\]

When \( f \) and \( g \) are PDFs for GMMs, the expression for \( f \) becomes (with an analogous expression for \( g \)):

\[
f(x) = \sum_{a=1}^{A} \omega_a^f f_a(x) = \sum_{a=1}^{A} \omega_a^f N \left( x; \mu_a^f, \Sigma_a^f \right)
\]

Durrieu and Thiran define the bounds for KL-Divergence between GMMs to be:

\[
D_{lower}(f||g) = \sum_a \omega_a^f \log \frac{\sum_a \omega_a^f e^{-D_{KL}(f_a||f_a)}}{\sum_b \omega_b^g f_{ab}} - \sum_a \omega_a^f H(f_a)
\]

\[
D_{upper}(f||g) = \sum_a \omega_a^f \log \frac{\sum_a \omega_a^g e^{-D_{KL}(f_a||g_a)}}{\sum_b \omega_b^g f_{ab}} + \sum_a \omega_a^f H(f_a)
\]
where \(H(f_a)\) is the entropy of \(f_a\), and the normalization constants of the product of the individual Gaussians are given by:

\[
\log t_{ab} = -\frac{d}{2} \log 2\pi - \frac{1}{2} \log |\Sigma_a^f + \Sigma_b^g| - \frac{1}{2} (\mu_b^g - \mu_a^f)^T (\Sigma_a^f + \Sigma_b^g)^{-1} (\mu_b^g - \mu_a^f) \tag{10}
\]

\[
\log z_{aa} = -\frac{d}{2} \log 2\pi - \frac{1}{2} \log |\Sigma_a^f + \Sigma_a^f| - \frac{1}{2} (\mu_a^f - \mu_a^f)^T (\Sigma_a^f + \Sigma_a^f)^{-1} (\mu_a^f - \mu_a^f) \tag{11}
\]

We will focus on optimizing the following average of the lower and upper bounds of the KL-Divergence between GMMs as it was shown to be a good estimate of the KL-Divergence between GMMs in Durrieu et al. (2012).

\[
D_{\text{avg}}(f||g) = \frac{1}{2} \left( D_{\text{lower}}(f||g) + D_{\text{upper}}(f||g) \right)
\]

\[
= \frac{1}{2} \sum_a \omega_a^f \log \frac{\sum_b \omega_b^g e^{-D_{KL}(f_a||f_b)}}{\sum_b \omega_b^g t_{ab}} - \frac{1}{2} \sum_a \omega_a^f H(f_a)
\]

\[
+ \frac{1}{2} \sum_a \omega_a^f \log \frac{\sum_b \omega_b^g z_{aa}}{\sum_b \omega_b^g H(f_b)} + \frac{1}{2} \sum_a \omega_a^f \log \frac{\sum_b \omega_b^g z_{aa}}{\sum_b \omega_b^g e^{-D_{KL}(f_a||g_b)}}
\]

\[
D_{\text{avg}}(f||g) = \frac{1}{2} \sum_a \omega_a^f \left[ \log \sum_b \omega_b^g e^{-D_{KL}(f_a||f_b)} + \log \sum_b \omega_b^g z_{aa} - \log \sum_b \omega_b^g t_{ab} - \log \sum_b \omega_b^g e^{-D_{KL}(f_a||g_b)} \right]
\]

If we assume that the data is mean-centered, the normalization constant \(t_{ab}\) becomes:

\[
\log t_{ab} = -\frac{d}{2} \log 2\pi - \frac{1}{2} \log |\Sigma_a^f + \Sigma_b^g| \quad t_{ab} = e^{-\frac{d}{2} \log 2\pi - \frac{1}{2} \log |\Sigma_a^f + \Sigma_b^g|}
\]

\[
t_{ab} = (2\pi)^{-\frac{d}{2}} |\Sigma_a^f + \Sigma_b^g|^{-\frac{1}{2}} \tag{12}
\]

Similarly, \(z_{aa} = (2\pi)^{-\frac{d}{2}} |\Sigma_a^f + \Sigma_a^f|^{-\frac{1}{2}}\).

Plugging this into (8), we get:

\[
D_{\text{avg}}(f||g) = \frac{1}{2} \sum_a \omega_a^f \left[ \log \sum_b \omega_b^g e^{-D_{KL}(f_a||f_b)} + \log \sum_a \omega_a^f (2\pi)^{-\frac{d}{2}} |\Sigma_a^f + \Sigma_a^f|^{-\frac{1}{2}} - \log \sum_b \omega_b^g (2\pi)^{-\frac{d}{2}} |\Sigma_a^f + \Sigma_b^g|^{-\frac{1}{2}} - \log \sum_b \omega_b^g e^{-D_{KL}(f_a||g_b)} \right]
\]

\[
= \frac{1}{2} \sum_a \omega_a^f \left[ \log \sum_b \omega_b^g e^{-D_{KL}(f_a||f_b)} + \frac{d \log 2\pi}{2} + \log \sum_a \frac{\omega_a^f}{\sqrt{|\Sigma_a^f + \Sigma_a^f|}} - \frac{d \log 2\pi}{2} - \log \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} - \log \sum_b \omega_b^g e^{-D_{KL}(f_a||g_b)} \right]
\]

\[
D_{\text{avg}}(f||g) = \frac{1}{2} \sum_a \omega_a^f \left[ \log \sum_a \frac{\omega_a^f}{\sqrt{|\Sigma_a^f + \Sigma_a^f|}} - \log \sum_b \omega_b^g e^{-D_{KL}(f_a||g_b)} - \log \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} \right] \tag{13}
\]
6. Distance correlation-KL divergence separability

In the form of the equation above, KL-divergence between Gaussian mixtures is separable into terms that only depend on the KL-divergence between the multivariate Gaussian components that form the mixture. We can substitute \(-D_{KL}(f_a||f_a)\) with distance correlation \(\text{DCor}(\Sigma_a^f, \Sigma_a^f)\) and \(-D_{KL}(f_a||g_b)\) with \(\text{DCor}(\Sigma_a^f, \Sigma_b^g)\) instead based on our Theorem 1 below which shows that optimizing KL-divergence between multivariate Gaussians is equivalent to optimizing distance correlation for the same.

**Theorem 1** Minimization of distance correlation \(\text{argmin}_Z \text{dCor}(X, Z)\) with respect to \(Z\) maximizes the Kullback-Leibler divergence, \(KL(X||Z)\) for \(X \sim \mathcal{N}(0, \Sigma_X)\) and \(Z \sim \mathcal{N}(0, \Sigma_Z)\)

**Proof** Distance correlation can be represented as \(\text{det}(K)\) by \(\text{Fischer’s inequality} \) for the upper bound and \(\text{Cauchy-Schwarz inequality} \) for the lower bound. For covariance matrices \(\Sigma_X = XTX\) and \(\Sigma_Z = ZTZ\) we have

\[
\text{det}[(XTX)^2] \text{det}([TZ^2]) \leq \text{Tr}(XTXZ^2)
\]

by arithmetic-geometric mean inequality for the lower bound and Cauchy-Schwarz inequality for the upper bound on distance covariance \(\text{Tr}(XTXZ^2)\). \(\log \det(ZTZ)\) is the differential entropy \(h(Z)\) up to a constant for multivariate Gaussians. Similarly, the joint entropy \(h(XZ)\) is given by \(\log \det(\Sigma)\) where \(\Sigma = \begin{bmatrix} XTX & XTZ \\ ZTX & ZTZ \end{bmatrix}\). Kullback-Leibler divergence is defined using joint entropy and entropy as \(h(XZ) = h(X,Z) - h(X)\). By Fischer’s inequality, we have

\[
\det(\Sigma) \leq \det(XTX) \det(ZTZ)
\]

As \(\det(XTX)\) is fixed and \(\det(ZTZ)\) decreases with decrease in distance covariance, an increase of \(h(XZ)\) is only possible when \(h(X,Z)\) increases which is itself only possible when \(\text{Tr}(XTX)\) decreases. Thereby minimizing sum of distance covariance and \(\text{Tr}(XTX)\) maximizes the Kullback-Leibler divergence in the direction stated above while it also minimizes differential entropy \(\det(ZTZ)\).

Therefore we have the following objective that needs to be optimized instead.

\[
\frac{1}{2} \sum_a \omega_a^f \left[ \log \sum_a \omega_a^f e^{\text{DCov}(\Sigma_a^f, \Sigma_a^f)} + \log \sum_a \omega_a^f \sqrt{\Sigma_a^f} \text{Det}^{1/2} \Sigma_a^f - \log \sum_b \omega_b^g e^{\text{DCov}(\Sigma_a^f, \Sigma_b^g)} - \log \sum_b \omega_b^g \sqrt{\Sigma_a^f + \Sigma_b^g} \right]
\]

Note that two terms are constant in here with respect to the target mixture distribution as follows

\[
D_{\text{avg}}(f||g) = \frac{1}{2} \sum_a \omega_a^f \left[ C_1 + C_2 - \log \sum_b \omega_b^g e^{\text{DCov}(\Sigma_a^f, \Sigma_b^g)} - \log \sum_b \omega_b^g \sqrt{\Sigma_a^f + \Sigma_b^g} \right]
\]
(a) Classifier reaches chance accuracy upon attempting to classify multiple real splinter samples from decoy splinters on the EEG eye state dataset for increasing number of decoy samples shown in the legend. The real splinter samples and decoy splinter samples were generated from the mixture distributions learnt by our scheme. In practice, there is only one sample per real distribution and multiple decoy samples, thereby making these results a lot more conservative.

(b) Classifier reaches chance accuracy upon attempting to classify multiple real splinter samples from decoy splinters on the Avila dataset for increasing number of decoy samples shown in the legend. The rest of setup is as described in figure a.)

(c) Classifier reaches chance accuracy upon attempting to classify multiple real splinter samples from decoy splinters on the skin segmentation dataset for increasing number of decoy samples shown in the legend. The rest of setup is as described in figure a.) and the dataset dimensions are presented in Table 1.

(d) We show decreasing values of KL-divergence between the real and decoy splinters with respect to the iterations of our scheme. The splinters were generated while ensuring they are decorrelated with side-information of labels, so that they do not leak that information when sent to the server as part of splintering. A cross-section of this effect of decorrelating with labels while ensuring real and decoy splinters are not distinguished is presented in Fig.3 below

Figure 1: In subfigures a.), b.) and c.), we show that state of the art tuned classification models such as XGBoost cannot distinguish between the real and decoy splinters generated by our scheme, thereby making it really hard for the attacker to estimate the pair of mixture distributions used to sample the real and decoy splinters. We also show convergence of reducing KL-divergence between the learnt Gaussian mixture and the target Gaussian mixture.
6.1 Required conditions for convexity

We now share some required conditions for the convexity of the key terms above. The function \( \log \sum_b \frac{\omega^g_b}{\sqrt{\Sigma^f_a + \Sigma^g_b}} \) is convex if

\[
\omega^g_b \sum_b \left( \frac{\omega^g_b}{\sqrt{\Sigma^f_a + \Sigma^g_b}} - \omega^g_b \right) \geq 0
\]

as this results in a positive semi-definite Hessian. This condition simplifies to requiring

\[
\sqrt{\Sigma^f_a + \Sigma^g_b} \leq \omega^g_b, \forall b
\]

By the arithmetic-geometric-mean (A.G.M) inequality we have,

\[
\prod_{k=1}^n \lambda_k \leq \left( \sum_{k=1}^n \lambda_k \right)^n
\]

Therefore \( \sum_b \sqrt{\Sigma^f_a + \Sigma^g_b} \leq \sum_b \left[ \frac{\text{Tr}(\Sigma^f_a + \Sigma^g_b)}{n} \right]^n \) This implies that if,

\[
\sum_b \text{Tr}(\Sigma^f_a + \Sigma^g_b) \leq n \sqrt{\omega^g_b}, \forall b
\]

then the condition for convexity \( \sum_b \sqrt{\Sigma^f_a + \Sigma^g_b} \leq n \sqrt{\omega^g_b}, \forall b \) will be satisfied.

6.2 Convexity of LogSumExp(DCOV)

The function \( \text{LogSumExp}(p) = \log(\sum_i (e^p_i)) \) is convex. We now show that the LogSumExp function \( \log \sum_b \omega^g_b e^{\text{DCov}(\Sigma^f_a, \Sigma^g_b)} \) is convex as well. In fact, \( \text{LogSumExp}(f(z)) \) happens to be convex for any convex function \( f(z) \) as shown below.

\[
\frac{\partial^2}{\partial z^2} \log \sum e^{f_i(z)} = \frac{\partial}{\partial z} \left[ \frac{\sum (e^{f_i(z)} \frac{\partial}{\partial z} f_i(z))}{\sum e^{f_i(z)}} \right]
\]

which is equal to

\[
\frac{\sum e^f_i \frac{\partial^2}{\partial z^2} f_i(z)}{\sum e^{f_i(z)}} + \frac{\sum e^f_i \left[ \frac{\partial}{\partial z} f_i(z) \right]^2}{\sum e^{f_i(z)}} - \frac{\left( \sum e^f_i \frac{\partial}{\partial z} f_i(z) \right)^2}{\left( \sum e^{f_i(z)} \right)^2}
\]

The first term is positive. The difference of the next two terms is positive due to Jensen’s inequality as

\[
\sum \left[ a_i \left( \frac{\partial}{\partial z} f_i(z) \right)^2 \right] \geq \left[ \sum a_i \frac{\partial}{\partial z} f_i(z) \right]^2
\]

This proves convexity of \( \log \sum_b \omega^g_b e^{\text{DCov}(\Sigma^f_a, \Sigma^g_b)} \).
<table>
<thead>
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<th>Dataset</th>
<th>Sample Size</th>
<th>Attributes</th>
<th>Balanced</th>
<th># of Classes</th>
</tr>
</thead>
<tbody>
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<td>14,980</td>
<td>15</td>
<td>Yes</td>
<td>2</td>
</tr>
<tr>
<td>Avila</td>
<td>20,867</td>
<td>10</td>
<td>Yes</td>
<td>12</td>
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<tr>
<td>Skin Segmentation</td>
<td>245,057</td>
<td>4</td>
<td>No</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 1: A listing of datasets that we used for empirical investigations is provided in this table along with their dimensions.

Figure 2: We show that the raw data of face image can only be reconstructed after using the right secret coefficients up to a high precision. We show in this qualitative example that the quality of reconstruction is lost if the precision is below 4 floating point decimal places as seen in the 2 reconstructed images on the top row.

Figure 3: The splinters were generated while ensuring they are decorrelated with side-information of labels, so that they do not leak that information when sent to the server as part of splintering. A cross-section of this effect of decorrelating with labels while ensuring real and decoy splinters are not distinguished. As seen the class labels of the raw data $X$ are originally separable while they are overlapping in the splinters. The splinters are always generated in a multivariate setting. This is a qualitative cross-sectional result on a single variable, for visualization purposes although the run was done on multivariate data.

6.3 Modified EM algorithm for our structured distribution learning problem

To maximize the average bound on KL-divergence $D_{\text{avg}}(f || g)$, the following has to be minimized

$$
\frac{1}{2} \sum_\alpha \omega^f_\alpha \left[ \log \sum_\alpha \omega^f_\alpha e^{D_{\text{Cov}}(\Sigma^f_\alpha, \Sigma^f_\alpha)} + \log \sum_\alpha \sqrt{\Sigma^f_\alpha + \Sigma^f_\alpha} - \log \sum_\beta \omega^g_\beta e^{D_{\text{Cov}}(\Sigma^g_\beta, \Sigma^g_\beta)} - \log \sum_\beta \sqrt{\Sigma^g_\beta + \Sigma^g_\beta} \right] - \log \sum_\beta \sqrt{\Sigma^g_\beta + \Sigma^g_\beta}.
$$

(19)
With \( \Sigma_g = \frac{1}{N-1}Z_b^T Z_b \), where \( N \) is the number of samples, our problem is equivalent to minimizing the following for each component \( a \)

\[
\omega_a f \log \sum_\alpha \omega_\alpha f \text{D Cov}(\Sigma_f, \Sigma_\alpha f) + \omega_a f \log \sum_\alpha \frac{\omega_\alpha f}{|\Sigma_a f + \Sigma_\alpha|}
\]

\[
- \omega_b g \log \sum_b \omega_b g \text{D Cov}(\Sigma_f, \frac{1}{N-1}Z_b^T Z_b) - \omega_a f \log \sum_b \frac{\omega_b g}{|\Sigma_a f + \Sigma_\alpha|}
\]

\[
= \omega_a (C_1 + C_2) - \omega_b \log \sum_b \omega_b g \text{D Cov}(\Sigma_f, \frac{1}{N-1}Z_b^T Z_b) - \omega_a f \log \sum_b \frac{\omega_b g}{|\Sigma_a f + \Sigma_\alpha|}
\]

\[
- \omega_b g \text{D Cov}(\Sigma_f, \frac{1}{N-1}(Z_b - \mu_b g)^T(Z_b - \mu_b g)) - \omega_a f \log \sum_b \frac{\omega_b g e^{-\frac{1}{2}(\mu_b g - \mu_a f)^T(\Sigma_a f + \frac{1}{N-1}(Z_b - \mu_b g)^T(Z_b - \mu_b g))^{-1}(\mu_b g - \mu_a f)}}{|\Sigma_a f + \frac{1}{N-1}(Z_b - \mu_b g)^T(Z_b - \mu_b g)|}
\]

\[
+ \omega_a f (C_1 + C_2) + \lambda \text{EMLoss}
\]

where the EMLoss in the last term is the standard EM loss. Here, the objective function is regularized with the standard loss used in EM-algorithms for estimating Gaussian mixtures. Therefore we now have a modified EM algorithm that learns Gaussian mixtures with respect to a target distribution while satisfying the closeness constraints with respect to KL-divergence.

**E-step updates:** For each component \( b \) at step \( t \), compute

\[
\gamma_{ib}^{(t+1)} = \frac{\omega_b^g p \left( y_i | \mu_b^g, \Sigma_b^g \right)}{\sum_{b'} B \omega_{b'}^g p \left( y_i | \mu_{b'}^g, \Sigma_{b'}^g \right)}, \quad i = 1, \ldots, N
\]

and finally

\[
n_b^{(t+1)} = \sum_{i=1}^N \gamma_{ib}^{(t+1)}
\]

**M-step updates:** For each component \( b \), compute the following update

\[
\omega_b^{g(t+1)} = \frac{n_b^{(t+1)}}{N}
\]

Using Powell minimization method we obtain the update for the mean vectors as follows
Using Powell minimization method for $Z_b$ we optimize

$$Z_b^{(t+1)} = \min_Z \left\{ -\omega_b^f \log \sum_{b' \neq b} \omega_{b'}^{g(t)} e^{-\frac{1}{2} \left( \mu_{b'}^{g(t)} - \mu_b^{g(t)} \right)^T \left( Z_{b'}^{(t)} - \mu_{b'}^{g(t)} \right) \left( Z_{b'}^{(t)} - \mu_{b'}^{g(t)} \right)^T} + \omega_b^{g(t)} e^{-\frac{1}{2} \left( \mu_b^{g(t)} - \mu_b^{g(t)} \right)^T \left( Z_b^{(t)} - \mu_b^{g(t)} \right) \left( Z_b^{(t)} - \mu_b^{g(t)} \right)^T} \right\} + \omega_b^f (C_1 + C_2)$$

$$+ \frac{1}{2} \sum_{b' \neq b} \gamma_{ib'}^{(t+1)} \text{Tr} \left( \left( \frac{1}{N-1} \left( Z_{b'}^{(t)} - \mu_{b'}^{g(t)} \right)^T \left( Z_{b'}^{(t)} - \mu_{b'}^{g(t)} \right) \right)^{-1} \left( Z_{b'}^{(t)} - \mu_{b'}^{g(t)} \right) \left( Z_{b'}^{(t)} - \mu_{b'}^{g(t)} \right)^T \right)$$

$$+ \frac{1}{2} \sum_{i=1}^N \gamma_{ib}^{(t+1)} \text{Tr} \left( \left( \frac{1}{N-1} \left( Z_b^{(t)} - \mu_b^{g(t)} \right)^T \left( Z_b^{(t)} - \mu_b^{g(t)} \right) \right)^{-1} \left( Z_b^{(t)} - \mu_b^{g(t)} \right) \left( Z_b^{(t)} - \mu_b^{g(t)} \right)^T \right).$$

### 7. Experiments

In this section, we share quantitative and qualitative empirical findings upon applying our proposed approach to various datasets described in Table 1.

In subfigures 1a), 1b) and 1c), we show that state of the art tuned classification model such as XGBoost cannot distinguish between the real and decoy splinters generated by our scheme, thereby making it really hard for the attacker to estimate the pair of mixture distributions used to sample the real and decoy splinters. We also show convergence of reducing KL-divergence between the learnt Gaussian mixture and the target Gaussian mixture in Figure 1d.)

We also visualize example splinters including the decoy and real non-decoy splinters along with their reconstructions in Figure 2 upon choosing the right secret coefficients albeit at varying number of floating point precisions. We see that in this particular example the real image cannot be reconstructed back unless the coefficients were correctly chosen up to 4 decimal point places. The
2 by 2 images of face reconstructions are for exactly chosen secret coefficients up to 2,3, 4 and 5 decimal places.

In Figure 3, we show that the splinters were generated while ensuring they are decorrelated with side-information of labels, so that they do not leak that information when sent to the server as part of splintering. A cross-section of this effect of decorrelating with labels while ensuring real and decoy splinters are not distinguished. As seen the class labels of the raw data $X$ are originally separable while they are overlapping in the splinters. The splinters are always generated in a multivariate setting. This is a qualitative cross-sectional result on a single variable, for visualization purposes although the run was done on multivariate data. We use default SciPy parameters for Powell minimization to optimize $\mu$ and parameters of $\text{ftol} = 0.001$, $\text{x_tol} = 0.001$, $\text{maxfev} = 4000$ for optimizing $Z_b$.

8. Conclusion

We provide a new scheme for secure computation called splintering that is well suited for distributed machine learning given its efficiency with respect to the resources of compute and bandwidth. We share various theoretical guarantees and practical insights of our approach. We would also like to extend this scheme to other useful private operations and thereby build a toolbox for splintering based computational pipelines. We find our approach to be particularly suitable for privatizing training and inference on split learning a popular distributed machine learning technique.

References


Cynthia Dwork, Frank McSherry, Kobbi Nissim, and Adam Smith. Differential privacya primer for the perplexed.. In *Conf. of European Statisticians, Joint UNECE/Eurostat work session on statistical data confidentiality*, 2011.


9. Appendix

Differentiating \( \log \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} \) with respect to \( \Sigma_b^g \),

\[
\frac{\partial}{\partial \Sigma_b^g} \left( \log \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} \right) = \frac{\omega_b^g}{\sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}}} \cdot \frac{\partial}{\partial \Sigma_b^g} \left( |\Sigma_a^f + \Sigma_b^g|^{-\frac{1}{2}} \right) = \frac{\omega_b^g}{\sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}}} \cdot \frac{1}{2} |\Sigma_a^f + \Sigma_b^g|^{-\frac{3}{2}} \cdot \frac{\partial}{\partial \Sigma_b^g} \left( |\Sigma_a^f + \Sigma_b^g| \right)
\]

The second derivative is then given by:

\[
\frac{\partial^2}{\partial (\Sigma_b^g)^2} \left( \log \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} \right) = \frac{\partial}{\partial \Sigma_b^g} \left( \frac{\omega_b^g}{2 |\Sigma_a^f + \Sigma_b^g|^\frac{3}{2}} \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} \right) \cdot \frac{\partial}{\partial \Sigma_b^g} \left( |\Sigma_a^f + \Sigma_b^g| \right)
\]

\[
= \frac{\omega_b^g}{2} \left( |\Sigma_a^f + \Sigma_b^g|^\frac{3}{2} \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} \right)^{-2} \cdot \frac{\partial}{\partial \Sigma_b^g} \left( |\Sigma_a^f + \Sigma_b^g|^\frac{3}{2} \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} \right) \cdot \frac{\partial}{\partial \Sigma_b^g} \left( |\Sigma_a^f + \Sigma_b^g| \right)
\]

\[
= \frac{\omega_b^g}{2} \left( |\Sigma_a^f + \Sigma_b^g|^\frac{3}{2} \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} \right)^{-2} \cdot \frac{\partial^2}{\partial (\Sigma_b^g)^2} \left( |\Sigma_a^f + \Sigma_b^g| \right)
\]

where

\[
\frac{\partial}{\partial \Sigma_b^g} \left( |\Sigma_a^f + \Sigma_b^g|^\frac{3}{2} \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} \right) = \frac{3}{2} |\Sigma_a^f + \Sigma_b^g|^\frac{1}{2} \cdot \frac{\partial}{\partial \Sigma_b^g} \left( |\Sigma_a^f + \Sigma_b^g| \right) \cdot \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} + |\Sigma_a^f + \Sigma_b^g|^\frac{3}{2} \cdot \frac{\partial}{\partial \Sigma_b^g} \left( |\Sigma_a^f + \Sigma_b^g| \right)
\]

\[
= \frac{1}{2} \cdot \frac{\partial}{\partial \Sigma_b^g} \left( |\Sigma_a^f + \Sigma_b^g| \right) \left( 3 |\Sigma_a^f + \Sigma_b^g|^\frac{1}{2} \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} - \omega_b^g \right)
\]
Then, for the second derivative, we have

$$
\frac{\partial^2}{\partial (\Sigma_b^g)^2} \left( \log \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a + \Sigma_b^g|}} \right)
$$

$$
= \omega_b^g \left[ \frac{\left( \partial \Sigma_a^f / \partial \Sigma_b^g \right) \Sigma_a^f + \Sigma_b^g}{\Sigma_a^f + \Sigma_b^g} \right] \left[ \frac{\left( 3 \Sigma_a^f + \Sigma_b^g \right) \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a + \Sigma_b^g|}} - \omega_b^g}{\Sigma_a^f + \Sigma_b^g} \right] - \frac{\partial^2}{\partial (\Sigma_b^g)^2} \left( |\Sigma_a^f + \Sigma_b^g| \right)
$$

Using the property $\nabla_X \det^k(X + tY) = k \det^k(X + tY)(X + tY)^{-T}$ (from page 449 of Convex Optimization & Euclidean Distance Geometry) with $k = t = 1$, the first and second derivatives of $|\Sigma_a^f + \Sigma_b^g|$ are given by

$$
\frac{\partial}{\partial \Sigma_b^g} \left( |\Sigma_a^f + \Sigma_b^g| \right) = \Sigma_a^f + \Sigma_b^g \left( (\Sigma_a^f + \Sigma_b^g)^{-T} \right)^{-1} = \Sigma_a^f + \Sigma_b^g \left( (\Sigma_a^f + \Sigma_b^g)^{-1} \right)^{-1}
$$

Note that $\left( \Sigma_a^f + \Sigma_b^g \right)$ must be invertible.

$$
\frac{\partial^2}{\partial (\Sigma_b^g)^2} \left( \Sigma_a^f + \Sigma_b^g \right) = \frac{\partial}{\partial \Sigma_b^g} \left( \left( \Sigma_a^f + \Sigma_b^g \right) \left( \Sigma_a^f + \Sigma_b^g \right)^{-1} \right)
$$

Applying Property 40 of the Matrix Cookbook, $\partial(X^{-1}) = -X^{-1}(\partial X)X^{-1}$, and equation (12):

$$
\frac{\partial^2}{\partial (\Sigma_b^g)^2} \left( \Sigma_a^f + \Sigma_b^g \right) = \Sigma_a^f + \Sigma_b^g \left[ \left( \Sigma_a^f + \Sigma_b^g \right)^{-1} \right] - \left( \Sigma_a^f + \Sigma_b^g \right)^{-1} \partial \Sigma_b^g \left( \Sigma_a^f + \Sigma_b^g \right) \left( \Sigma_a^f + \Sigma_b^g \right)^{-1}
$$

If $\partial \Sigma_b^g \left( \Sigma_a^f + \Sigma_b^g \right) = I,$

$$
\frac{\partial^2}{\partial (\Sigma_b^g)^2} \left( \Sigma_a^f + \Sigma_b^g \right) = \Sigma_a^f + \Sigma_b^g \left[ \left( \Sigma_a^f + \Sigma_b^g \right)^{-1} \right] - \left( \Sigma_a^f + \Sigma_b^g \right)^{-1} \cdot I \cdot \left( \Sigma_a^f + \Sigma_b^g \right)^{-1}
$$

$$
= \Sigma_a^f + \Sigma_b^g \left[ \left( \Sigma_a^f + \Sigma_b^g \right)^{-1} \right] - \left( \Sigma_a^f + \Sigma_b^g \right)^{-1} \cdot \left( \Sigma_a^f + \Sigma_b^g \right)^{-1}
$$

$$
= \Sigma_a^f + \Sigma_b^g \left[ \left( \Sigma_a^f + \Sigma_b^g \right)^{-1} \right] - \left( \Sigma_a^f + \Sigma_b^g \right)^{-1} \cdot \left( \Sigma_a^f + \Sigma_b^g \right)^{-1}
$$

$$
= 0
$$

(25)
Therefore,

\[
\frac{\partial^2}{\partial (\Sigma_b^g)^2} \left( \log \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} \right)
= \frac{\omega_b^g}{2 \left| \Sigma_a^f + \Sigma_b^g \right|^{\frac{3}{2}} \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} \left[ \left( \left| \Sigma_a^f + \Sigma_b^g \right| \left( \Sigma_a^f + \Sigma_b^g \right)^{-1} \right)^2 \left( 3 \left| \Sigma_a^f + \Sigma_b^g \right|^{\frac{1}{2}} \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} - \omega_b^g \right) \right] - 0 \right)
= \frac{\omega_b^g}{2 \left| \Sigma_a^f + \Sigma_b^g \right|^{\frac{3}{2}} \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} \left[ \left( \left| \Sigma_a^f + \Sigma_b^g \right| \left( \Sigma_a^f + \Sigma_b^g \right)^{-1} \right)^2 \left( 3 \left| \Sigma_a^f + \Sigma_b^g \right|^{\frac{1}{2}} \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} - \omega_b^g \right) \right] \cdot \frac{\left( \left| \Sigma_a^f + \Sigma_b^g \right| \left( \Sigma_a^f + \Sigma_b^g \right)^{-1} \right)^2 \left( 3 \left| \Sigma_a^f + \Sigma_b^g \right|^{\frac{1}{2}} \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} - \omega_b^g \right)}{4 \left| \Sigma_a^f + \Sigma_b^g \right| \left( \sum_b \frac{\omega_b^g}{\sqrt{|\Sigma_a^f + \Sigma_b^g|}} \right)^2}
\]